(6TH SEMESTER)

ELECTROMAGNETIC THEORY (3-1-0)

MODULE-I (10 HOURS)

Representation of vectors in Cartesian, Cylindrical and Spherical coordinate system, Vector products, Coordinate transformation.

The Law of force between elementary electric Charges, Electric Field Intensity and Potential due to various charge configuration, Electric Flux density, Gauss law and its application, Application of Gauss Law to differential Volume element, Divergence Theorem. Potential Gradient, Dipole, and Energy Density in Electrostatic Field.

MODULE-II (10 HOURS)

Current and Conductors, Continuity of Current, Conductor Properties and Boundary Conditions. The Method of Images, Nature of dielectric Materials, Boundary Conditions for Perfect Dielectric Materials Capacitance, Poisson's & Laplace equation, Uniqueness Theorem, Analytical Solution in one dimension.- Use of MATLAB

Steady Magnetic Field: Biot Savart Law, Ampere's Circuital Law, Stoke's Theorem, Scalar and Vector Magnetic Potential,

MODULE-III (10 HOURS)

Force on a moving Charge, Force on a differential Current Element, Force & Torque Magnetisation & Permeability, Magnetic Boundary Conditions, Inductance & Mutual Inductance.

Time Varying Fields: Faraday's Law, Displacement Current, Maxwell's Equation.

MODULE-IV (10 HOURS)

Wave propagation in Free Space, Dielectric, and Good Conductor. Poynting's Theorem and wave power, Wave polarization, Reflection and Transmission of Uniform Plane Waves at Normal & Oblique incidence, Standing Wave Ratio, Basic Wave Guide Operation and Basic Antenna Principles.

BOOKS

[1]. W. H. Hayt (Jr), J. A. Buck, "Engineering Electromagnetics", TMH

[2]. K. E. Lonngren, S.V. Savor, "Fundamentals of Electromagnetics with Matlab", PHI

[3]. E.C.Jordan, K.G. Balmain, "Electromagnetic Waves & Radiating System", PHI.

[4]. M. N. Sadiku, "Elements of Electromagnetics", Oxford University Press.

MODULE-I

INTRODUCTION:

Electromagnetic theory is concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering. Electromagnetic theory is also required for the understanding, analysis and design of various electrical, electromechanical and electronic systems.

Electromagnetic theory can be thought of as generalization of circuit theory. Electromagnetic theory deals directly with the electric and magnetic field vectors where as circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively.

Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is the required mathematical tool with which electromagnetic concepts can be conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory is prerequisite, first we will go through vector algebra.

Applications of Electromagnetic theory:

This subject basically consist of static electric fields, static magnetic fields, time-varying fields & it's applications.

One of the most common applications of electrostatic fields is the deflection of a charged particle such as an electron or proton in order to control it's trajectory. The deflection is achieved by maintaining a potential difference between a pair of parallel plates. This principle is used in CROs, ink-jet printer etc. Electrostatic fields are also used for sorting of minerals for example in ore separation. Other applications are in electrostatic generator and electrostatic voltmeter.

The most common applications of static magnetic fields are in dc machines. Other applications include magnetic deflection, magnetic separator, cyclotron, hall effect sensors, magneto hydrodynamic generator etc.

Vector Analysis:

The quantities that we deal in electromagnetic theory may be either **scalar** or **vectors**. Scalars are quantities characterized by magnitude only. A quantity that has direction as well as magnitude is called a vector. In electromagnetic theory both scalar and vector quantities are function of *time* and *position*.

A vector $\vec{A}$ can be written as, $\vec{A} = \hat{a} A$, where, $A = |\vec{A}|$ is the magnitude and $\hat{a} = \frac{\vec{A}}{|A|}$ is the unit vector which has unit magnitude and same direction as that of $\vec{A}$.

Two vector $\vec{A}$ and $\vec{B}$ are added together to give another vector $\vec{C}$. We have

$$\vec{C} = \vec{A} + \vec{B} \quad ....(1.1)$$

Let us see the animations in the next pages for the addition of two vectors, which has two rules:

1: Parallelogram law and **2: Head & tail rule**

Scaling of a vector is defined as $\vec{C} = \alpha \vec{B}$, where $\vec{C}$ is scaled version of vector $\vec{B}$ and α is a scalar. Some important laws of vector algebra are:

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ Commutative Law..(1.3)

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ Associative Law...(1.4)

$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$ Distributive Law ....(1.5)

The position vector $\vec{r_Q}$ of a point *P* is the directed distance from the origin (*O*) to *P*, i.e., $\vec{r_Q}$ = $\overrightarrow{OP}$.

If $\vec{r_Q}$ = OP and $\vec{r_P}$ = OQ are the position vectors of the points P and Q then the distance vector

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r_P} - \vec{r_Q}$$

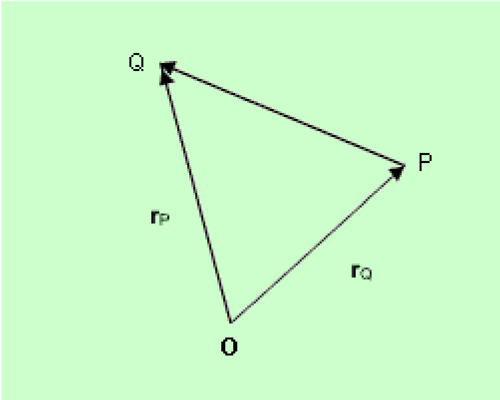


Fig 1.3: Distance Vector

Product of Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are multiplied, the result is either a scalar or a vector depending how the two vectors were multiplied. The two types of vector multiplication are:

Scalar product (or dot product) $\vec{A} \cdot \vec{B}$ gives a scalar.

Vector product (or cross product) $\vec{A} \times \vec{B}$ gives a vector.

The dot product between two vectors is defined as $\vec{A} \cdot \vec{B} = |A||B| cos\theta_{AB}$ ....(1.6)

Vector product $\vec{A} \times \vec{B} = |A||B| \sin \theta_{AB} \cdot \vec{n}$

$\vec{n}$ is unit vector perpendicular to $\vec{A}$ and $\vec{B}$

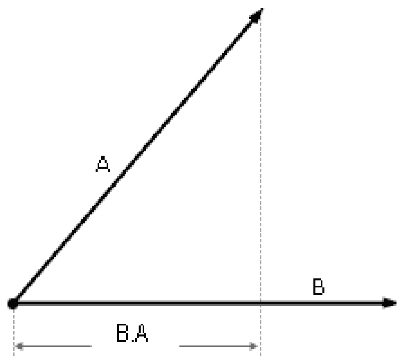


Fig 1.4 : Vector dot product

The dot product is commutative i.e., $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$ and distributive i.e., $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$.
Associative law does not apply to scalar product.

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$. $\vec{A}\times\vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, the magnitude is given by $|A||B|\sin\theta_{AB}$ and direction is given by right hand rule.

$$\vec{A}\times\vec{B}=\hat{a}_n\, AB\sin\theta_{AB} \quad \text{.......(1.7)}$$

where $\hat{a}_n$ is the unit vector given by,

$$\hat{a}_n=\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}.$$

The following relations hold for vector product.

$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$ i.e., cross product is non commutative ....(1.8)

$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$ i.e., cross product is distributive.......................(1.9)

$\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$ i.e., cross product is non associative..............(1.10)

Scalar and vector triple product :

Scalar triple product $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)=\vec{C}\cdot\left(\vec{A}\times\vec{B}\right)$ ....(1.11)

Vector triple product $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$ ....(1.12)

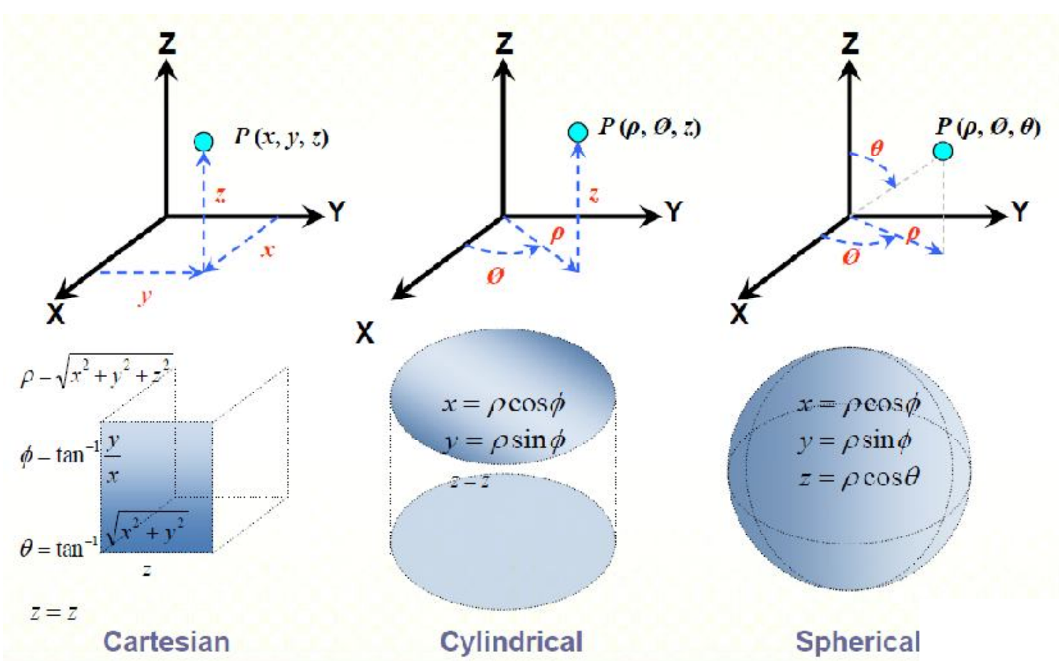
Co-ordinate Systems:

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in an orthogonal coordinate system. An orthogonal system is one in which the co-ordinates are mutually perpendicular.

In electromagnetic theory many physical quantities are vectors, which are having different components. So we use orthogonal co-ordinate systems for representing those quantities and depending on the symmetry of the physical quantities different coordinate systems are used.

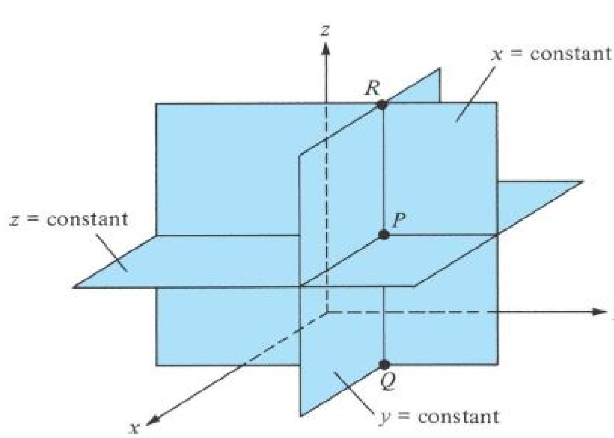
The electromagnetic (static) problem cases

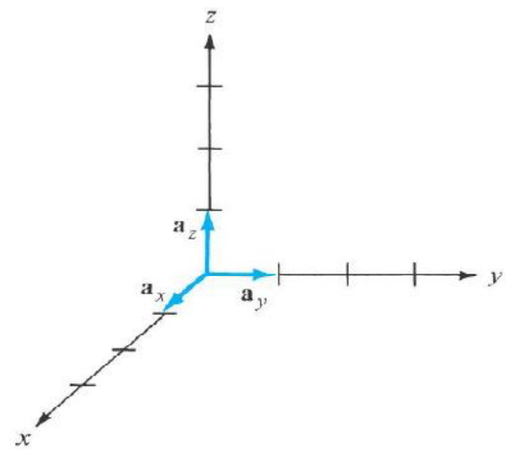
Coordinate	The problems most often encountered	
	Electrostatic Cases	Magnetostatic Cases
Cartesian	- Any uniform charge distribution using a scale of cartesian coordinates.	- The current flowing on the horizontal infinite extent plane.
Cylindrical	- Uniform charge distribution on cylindrical conductor. - Uniform charge distribution on the infinite length of line. - Uniform charge distribution on the infinite extent horizontal plane. - Uniform charge distribution on the horizontal circle plane. - Uniform charge distribution on the circle line.	- The current flowing in circumference. - The current flowing in an infinite straight line. - The current flows in the solenoid and toroid.
Spherical	- Uniform charge distribution on the surface of a sphere. - Uniform charge distribution of the point.	- Problem for the case of spheres less found.



Cartesian Co-ordinate System :

A point $P(x, y, z)$ in Cartesian co-ordinate system is represented as intersection of three planes x = constant, y = constant and z = constant, as shown in the figure below. The unit vectors along the three axes are as shown in the figure.





Coordinate system represented by (x,y,z) that are three orthogonal vectors in straight lines that intersect at a single point (the origin). The range of variation along the three axes are shown below.

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

The vector A in this coordinate system can be written as, $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

The differential lengths, area and volumes are as shown below.

- Differential Displacement (dl)

$$d\vec{l} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z$$

- Differential Surface Area (dS)

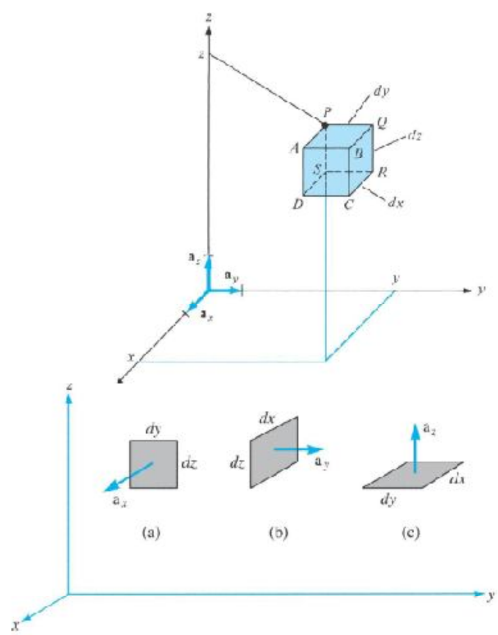
$$d\vec{S} = dydz\hat{a}_x$$

$$d\vec{S} = dxdz\hat{a}_y$$

$$d\vec{S} = dxdy\hat{a}_z$$

- Volume Differential (dv)

$$dv = dxdydz$$



Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u, v, w) = (r, \phi, z)$ as shown in figure below.

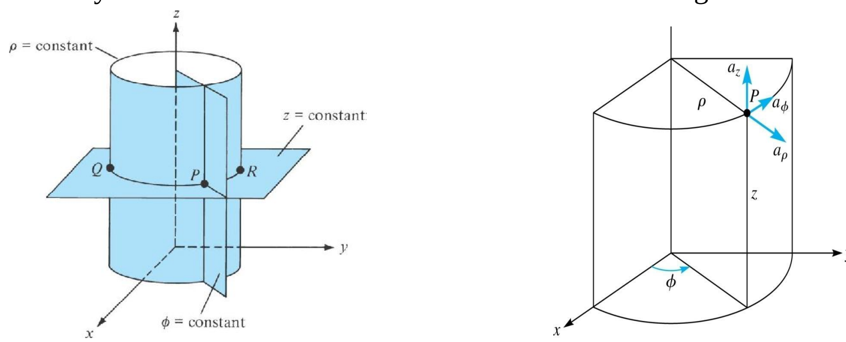


Figure: Cylindrical Coordinate System

Cylindrical Coordinate system represented by (ρ , ϕ ,z) that are three orthogonal vectors, varies in the range,

$$0 \le \rho < \infty$$

$$0 \le \phi < 2\pi$$

$$-\infty < z < \infty$$

The vector A in this coordinate system can be written as,

$$\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$$

The following equations can be used to convert between cylindrical and Cartesian coordinate systems,

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi$$

$$\phi = \tan^{-1}\frac{y}{x} \qquad y = \rho\sin\phi$$

$$z = z \qquad z = z$$

The differential elements in cylindrical coordinate system are shown below.

- Differential Displacement (dl)

$$d\vec{l} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z$$

- Differential Surface Area (dS)

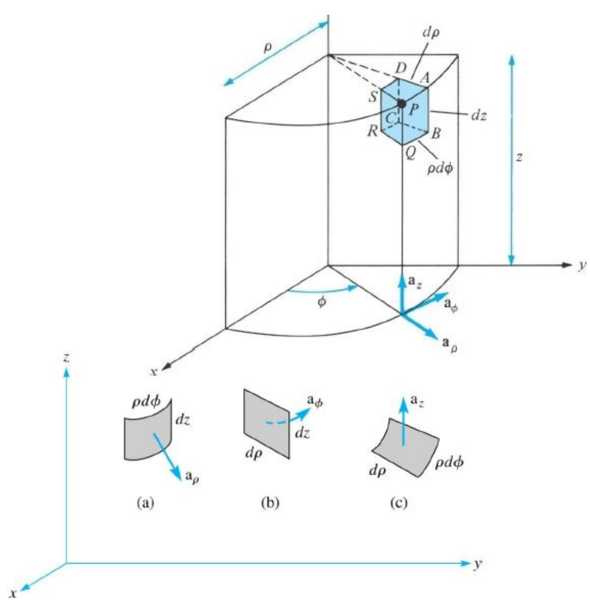
$$d\vec{S} = \rho d\phi dz\hat{a}_\rho$$

$$d\vec{S} = d\rho dz\hat{a}_\phi$$

$$d\vec{S} = \rho d\rho d\phi\hat{a}_z$$

- Volume Differential (dv)

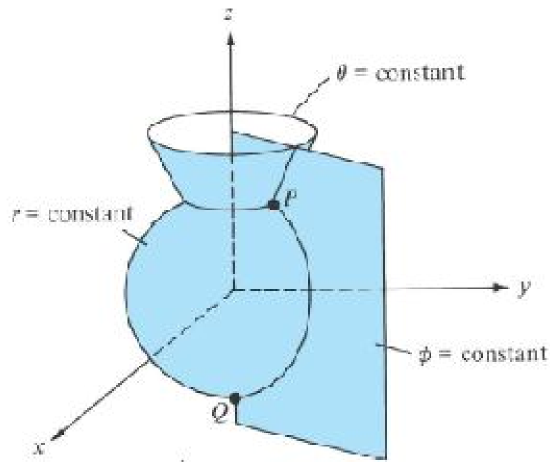
$$dv = \rho d\rho d\phi dz$$



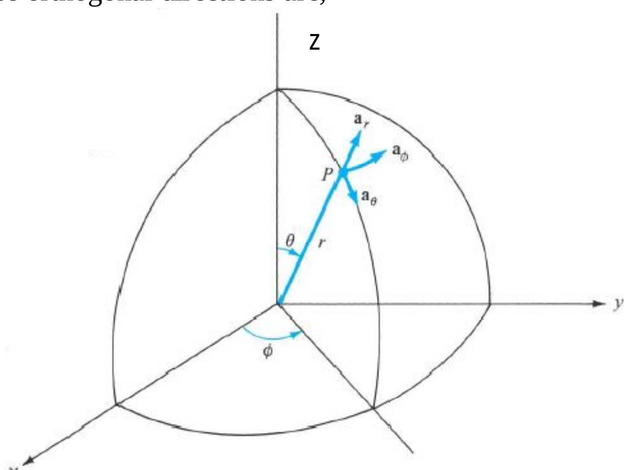
Spherical co-ordinate system:

Coordinate system represented by (r, θ , ϕ) that are three orthogonal vectors (as shown in the figure below) emanating from or revolving around the origin in the range,

$$0 \le r < \infty$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$



The unit vectors in the three orthogonal directions are,



The vector **A** in this coordinate system can be written as,

$$\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$$

The following equations can be used to convert between spherical and Cartesian coordinate systems.

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\phi$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad y = r\sin\theta\sin\phi$$
$$\phi = \tan^{-1}\frac{y}{x} \qquad z = r\cos\theta$$

The differential elements in spherical coordinate system are shown below.

- Differential Displacement (dl)

$$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi$$

- Differential Surface Area (dS)

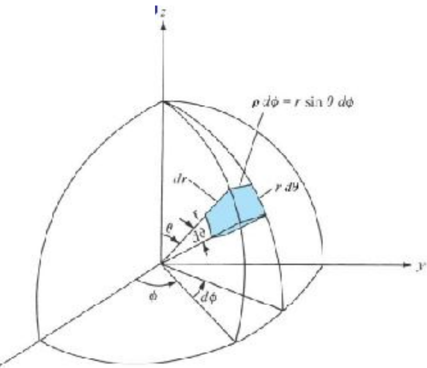
$$d\vec{S} = r^2 \sin\theta d\theta d\phi \hat{a}_r$$

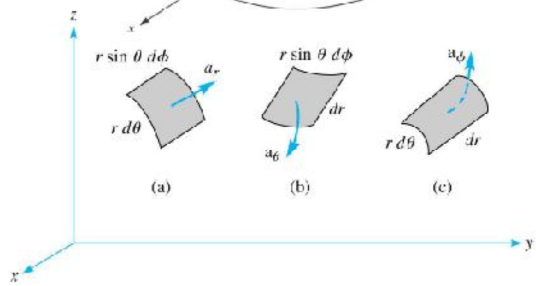
$$d\vec{S} = r\sin\theta dr d\phi \hat{a}_\theta$$

$$d\vec{S} = r dr d\theta \hat{a}_\phi$$

- Volume Differential (dv)

$$dv = r^2 \sin\theta dr d\theta d\phi$$



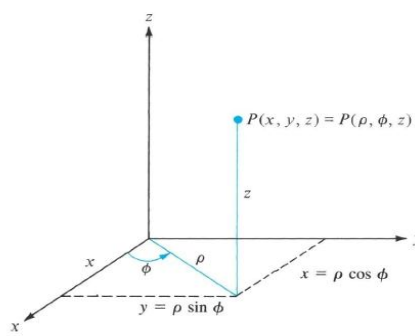


Co-ordinate transformation:

Matrix Transformations: Cartesian to Cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

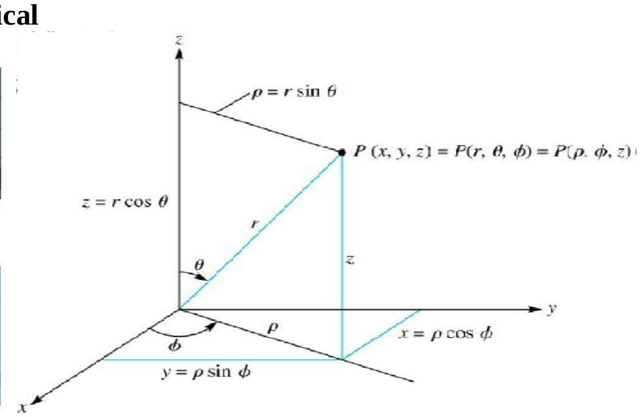
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$



Matrix Transformations: Cartesian to Spherical

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$



Del operator:
Del is a vector differential operator. The del operator will be used in for differential operations throughout any course on field theory. The following equation is the del operator for different coordinate systems.

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z = \nabla_{x,y,z}$$

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

Gradient of a Scalar:

• The gradient of a scalar field, V, is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z = \nabla V_{x,y,z}$$

• To help visualize this concept, take for example a topographical map. Lines on the map represent equal magnitudes of the scalar field. The gradient vector crosses map at the location where the lines packed into the most dense space and perpendicular (or normal) to them. The orientation (up or down) of the gradient vector is such that the field is increased in magnitude along that direction.

-Fundamental properties of the gradient of a scalar field

– The magnitude of gradient equals the maximum rate of change in V per unit distance

– Gradient points in the direction of the maximum rate of change in V

– Gradient at any point is perpendicular to the constant V surface that passes through that point

– The projection of the gradient in the direction of the unit vector **a**, is

$$\nabla V \cdot \hat{a}$$

and is called the directional derivative of V along **a**. This is the rate of change of V in the direction of **a**.

– If **A** is the gradient of V, then V is said to be the scalar potential of **A.**

Divergence of a Vector:
• The divergence of a vector, **A**, at any given point P is the outward flux per unit volume as volume shrinks about P.

$$div\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{S}}{\Delta v}$$

Divergence Theorem:
• The divergence theorem states that the total outward flux of a vector field, **A**, through the closed surface, S, is the same as the volume integral of the divergence of **A**.

• This theorem is easily shown from the equation for the divergence of a vector field.

$$\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$$

$$div\vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot d\vec{S}}{\Delta v}$$

$$\int_v \nabla \cdot \vec{A} dv = \oint_s \vec{A} \cdot d\vec{S}$$

Curl of a Vector:

• The curl of a vector, **A** is an axial vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

-Curl of a vector in each of the three primary coordinate systems are,

Cartesian
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z$$

Cylindrical
$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{a}_\rho - \left[\frac{\partial A_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z}\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{a}_z$$

Spherical
$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\hat{a}_r - \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi}\right]\hat{a}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{a}_\phi$$

Stokes Theorem:

• Stokes theorem states that the circulation of a vector field A, around a closed path, L is equal to the surface integral of the curl of A over the open surface S bounded by L. This theorem has been proven to hold as long as **A** and the curl of **A** are continuous along the closed surface S of a closed path L

• This theorem is easily shown from the equation for the curl of a vector field.

$$\vec{A} = A_1\hat{a}_1 + A_2\hat{a}_2 + A_3\hat{a}_3$$

$$curl\vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot d\vec{l}}{\Delta S}\right)_{max} \hat{a}_n$$

$$\oint_L \vec{A} \cdot d\vec{l} = \oint_s (\nabla \times \vec{A}) \cdot d\vec{S}$$

Classification of vector field:

The vector field, **A**, is said to be divergenceless (or solenoidal) if $\nabla \cdot \vec{A} = 0$.
– Such fields have no source or sink of flux, thus all the vector field lines entering an enclosed surface, S, must also leave it.

– Examples include magnetic fields, conduction current density under steady state, and imcompressible fluids

– The following equations are commonly utilized to solve divergenceless field problems

$$\nabla \cdot \vec{A} = 0$$

$$\oint_S \vec{A} \cdot d\vec{S} = \int_v \left(\nabla \cdot \vec{A}\right) dv = 0$$

$$\vec{F} = \nabla \times \vec{A}$$

• The vector field, **A**, is said to be potential (or irrotational) if $\nabla \times \vec{A} = 0$
– Such fields are said to be conservative. Examples include gravity, and electrostatic fields.

– The following equations are commonly used to solve potential field problems;

$$\nabla \times \nabla V = 0 \qquad \oint_L \vec{A} \cdot d\vec{l} = \oint_S \left(\nabla \times \vec{A}\right) \cdot d\vec{S} = 0$$

$$\nabla \times \vec{A} = 0 \qquad \vec{A} = -\nabla V$$

Solved Examples:

1. Given that $\boldsymbol{A} = \hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$

$$\boldsymbol{B} = -2\hat{a}_x + \hat{a}_y$$

   a. Determine the angle between the vectors ***A*** and ***B*** .

   b. Find the unit vector which is perpendicular to both ***A*** and ***B*** .

Solution:

a. We know that for two given vector A and B,

$$\boldsymbol{A}.\boldsymbol{B} = |\boldsymbol{A}|.|B|\cos\theta$$

For the two vectors A and B

$$|\boldsymbol{A}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\boldsymbol{B}| = \sqrt{(-2)^2 + 1} = \sqrt{5}$$

$\therefore \sqrt{14}.\sqrt{5}\cos\theta = 1$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{70}}\right)$

b. We know that is perpendicular to both A and B.

$$\therefore \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 2 & 3 \\ -2 & 0 & 1 \end{vmatrix}$$

$$= 2\hat{a}_x - (1+6)\hat{a}_4 + 4\hat{a}_3$$

The unit vector $\hat{n}$ perpendicular to both ***A*** and ***B*** is given by,

$$\hat{n} = \frac{\boldsymbol{A}\times\boldsymbol{B}}{|\boldsymbol{A}\times\boldsymbol{B}|} = \frac{2\hat{a}_x - 7\hat{a}_y + 4\hat{a}_z}{\sqrt{2^2 + (-7)^2 + 4^2}} = \frac{2\hat{a}_x - 7\hat{a}_y + 4\hat{a}_z}{\sqrt{69}}$$

2. Given the vectors $\boldsymbol{A} = \hat{a}_x + 2\hat{a}_y + 5\hat{a}_z$

$$\boldsymbol{B} = 5\hat{a}_\rho - \hat{a}_\phi + 3\hat{a}_z$$

Find :

a. The vector ***C*** = ***A*** + ***B*** at a point P (0, 2,-3).

b. The component of ***A*** along ***B*** at P.

Solution:

The vector B is cylindrical coordinates. This vector in Cartesian coordinate can be written as:

$$\boldsymbol{B} = B_x\hat{a}_x + B_y\hat{a}_y + B_z\hat{a}_z$$

Where $B_x = \boldsymbol{B}.\hat{a}_x = 5\hat{a}_\rho.\hat{a}_x - \hat{a}_\phi.\hat{a}_x + 3\hat{a}_3.\hat{a}_x$

$$= 5\cos\phi + \sin\phi$$

$$B_y = \boldsymbol{B}.\hat{a}_y = 5\sin\phi - \cos\phi$$

$$B_z = \mathbf{B}.\hat{a}_z$$

The point P(0,2,-3) is in the y-z plane for which $\phi = \frac{\pi}{2}$.

$$\therefore \mathbf{B} = \hat{a}_x + 5\hat{a}_y + 3\hat{a}_z$$

a. ***C* = *A* + *B***

$= \left(\hat{a}x + 2\hat{a}_y + 5\hat{a}_z\right) + \left(\hat{a}x + 5\hat{a}_y + 3\hat{a}_z\right)$

$= 2\hat{a}x + 7\hat{a}_y + 8\hat{a}_z$

b. Component of ***A*** along ***B*** is $|A|\cos\theta$ θ where is the angle between A is and B.

i.e., $\frac{\mathbf{A}.\mathbf{B}}{|B|} = \frac{1+10+15}{\sqrt{1+25+9}} = \frac{26}{\sqrt{35}}$

3. A vector field is given by

$$\mathbf{A} = \rho\cos\phi\,\hat{a}_\rho + \rho z\sin\phi\,\hat{a}_z$$

Transform this vector into rectangular co-ordinates and calculate its magnitude at P(1,0,1).

Solution:

Given, $\mathbf{A} = \rho\cos\phi\,\hat{a}_\rho + \rho z\sin\phi\,\hat{a}_z$

The components of the vector in Cartesian coordinates can be computed as follows:

$$A_x = \mathbf{A}.\hat{a}_x$$

$$= \rho\cos\phi\cos\phi = \rho\cos^2\phi = \sqrt{x^2+y^2}.\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{x^2}{\sqrt{x^2+y^2}}$$

$$A_y = \boldsymbol{A}\,\hat{a}_y$$

$$= \rho\cos\phi\sin\phi = \sqrt{x^2+y^2}.\frac{x}{\sqrt{x^2+y^2}}.\frac{y}{\sqrt{x^2+y^2}}$$

$$= \frac{xy}{\sqrt{x^2+y^2}}$$

$$A_z = \sqrt{x^2+y^2}.z\frac{y}{\sqrt{x^2+y^2}} = yz$$

$$\therefore \boldsymbol{A} = \frac{x^2}{\sqrt{x^2+y^2}}\hat{a}_x + \frac{xy}{\sqrt{x^2+y^2}}\hat{a}_y + yz\,\hat{a}_z$$

$$\boldsymbol{A}\big|_{(1,0,1)} = \frac{1}{\sqrt{1}}\hat{a}_x + 0 + 0$$

$$= \hat{a}_x$$

$$\therefore \left|\boldsymbol{A}\right|_{(1,0,1)} = 1$$

6. A given vector function is defined by $\boldsymbol{F} = y\hat{a}_x + x\hat{a}_y$. Evaluate the scalar line integral from a point P1(1, 1, -1) to P2(2, 4, -1).

   a. along the parabola $y = x^2$

   b. along the line joining the two points.

   Is ***F*** a conservative field?

Solution:

$$\boldsymbol{F} = y\hat{a}_x + x\hat{a}_y$$

$$d\boldsymbol{l} = dx\,\hat{a}_x + dy\,\hat{a}_y$$

$$\therefore \boldsymbol{F}.d\boldsymbol{l} = ydx + xdy$$

a. For evaluating the line integral along the parabola $y = x^2$, we find that

$$\mathrm{d}y = 2x\,\mathrm{d}x$$

$$\therefore \int_{P_1}^{P_2} F.\mathrm{d}l = \int_1^2 x^2\mathrm{d}x + 2x^2\mathrm{d}x$$

$$= \int_1^2 3x^2dx = \left[x^3\right]_1^2 = z$$

b. In this case we observe that z1 = z2 = -1, hence the line joining the points P1 and P2 lies in the z = -1 plane and can be represented by the equation

$$y - 1 = \frac{4-1}{2-1}(x-1)$$

Or, y = 3x -2

$\therefore$ d y = 3 d x

$\boldsymbol{F}$. d l = (3x -2)d x + x . 3d x

= (6x -2)d x

$$\therefore \int_{P_1}^{P_2} \boldsymbol{F}.\mathrm{d}\boldsymbol{l} = \int_1^2 (6x-2)\,\mathrm{d}x$$

$$= \left[\frac{6x^2}{2} - 2x\right]_1^2$$

$$= [12 - 4 - 3 + 2]$$

= 7

The field ***F*** is a conservative field.

7. If $\boldsymbol{D} = \frac{1}{r}\hat{a}_r$, calculate $\int_S \boldsymbol{D}.d\boldsymbol{S}$ over a hemispherical surface bounded by r =2 & $0 \le \theta \le \pi/2$.

Solution:

In spherical polar coordinates

$$dS = r^2 \sin\theta d\theta d\phi \hat{a}_r$$

$$\therefore \int_b D.dS = \int_0^2 \int_0^{\pi/2} 2.\sin\theta d\theta d\phi$$

$$= 4\pi \int_0^{2\pi} \sin\theta d\theta$$

$$= 4\pi$$

QUESTIONS:

(Preliminary Questions)

1.A small sphere of radius r and charge q is enclosed by a spherical shell of radius R and charge Q. Show that if q is positive, charhe q will necessarily flow from the sphere to the shell (when the two are connected by a wire) no matter what the charge Q on the shell is.(NCERT PHYSICS).

2.There are three concentric and conducting spheres of radius R, 2R and 4R respectively. Innermost sphere A and the outermost sphere C are coonected by a conducting wires while the intermediate sphere is uniformly charged to +Q. Find (a) charges on conductors A and C (b) potential of A and B. (c)If the spheres A and C are earthed.

3. If the vector field $T=(Axy+BZ^3)\, a_x+(3x^2-Cz)a_y+(3xz^2-y)a_z$ is irrotational,

determine A,B and C.

4. Express the divergence of a vector in rectangular, cylindrical and spherical coordinate system.

5.An electric field intensity is given as

$E = \frac{(100\cos\theta)}{r^3} a_r + \frac{(50Sin\theta)}{r^3} a_\theta$; Calculate the |E| and a unit vector in Cartesian coordinate in the direction of E at a point (r=2, $\theta = 60^0$, $\phi=20^0$)

Field:
A field is a function that specifies a particular physical quantity everywhere in a region. Depending upon the nature of the quantity under consideration, the field may be a vector or a scalar field. Example of scalar field is the electrostatic potential in a region while electric or magnetic fields at any point is the example of vector field.

Static Electric Fields:
Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges. The fundamental & experimentally proved laws of electrostatics are Coulomb's law & Gauss's theorem.

Coulomb's law & Electric field Intensity
Statement: The force between two point charges separated in vacuum or free space by a distance which is large compared to their size is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. It acts along the line joining the two charges.
Mathematically,

$$F = \frac{kQ_1Q_2}{R^2}$$

In SI units, Q_1 and Q_2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$, ε_0 is called the permittivity of free space & it's magnitude is

$$\varepsilon_0 = 8.854 \text{ x } 10^{-12} = \frac{1}{36\pi}10^{-9} \text{ F/m.}$$

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0\varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ ....(2.1)

As shown in the Figure 2.1 let the position vectors of the point charges $Q_1$and Q_2 are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\vec{F_{12}}$ represent the force on Q_1 due to charge Q_2.

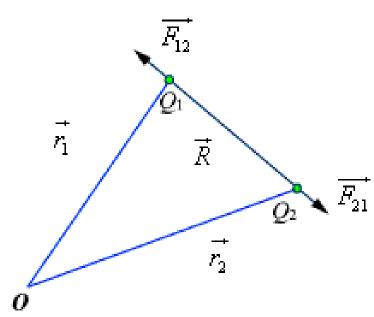


Fig 2.1: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$$\widehat{a_{12}} = \frac{\left(\vec{r_2} - \vec{r_1}\right)}{R} \quad \text{and} \quad \widehat{a_{21}} = \frac{\left(\vec{r_1} - \vec{r_2}\right)}{R} \quad \text{..................................(2.2)}$$

$\vec{F_{12}}$ can be defined as $\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{\left(\vec{r_2} - \vec{r_1}\right)}{\left|\vec{r_2} - \vec{r_1}\right|^3}$.

Similarly, the force on Q_1 due to charge Q_2 can be calculated and if $\vec{F_{21}}$ represents this force then we can write $\vec{F_{21}} = -\vec{F_{12}}$.

Suppose a charge q is placed in the vicinity of three other charges, q_1, q_2, and q_3, as is shown in Figure 2.2. Coulomb's law can be used to calculate the electric force between q and q_1, between q and q_2, and between q and q_3. Experiments have shown that the total force exerted by q_1, q_2 and q_3 on q is the vector sum of the individual forces, as shown in the equation below;

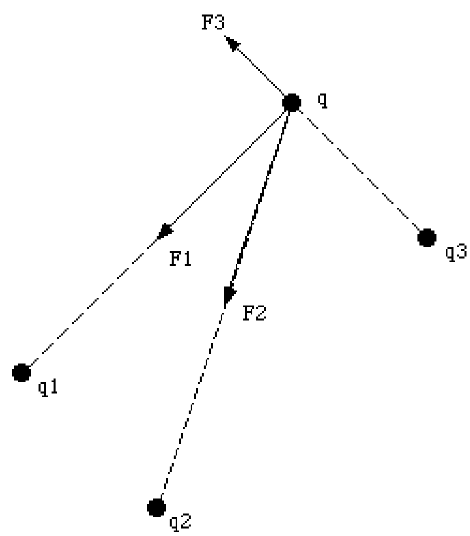


Figure 2.2. Superposition of electric forces.

$$\bar{F}_t = \bar{F}_{q_1 q} + \bar{F}_{q_2 q} + \bar{F}_{q_3 q} = \frac{q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3\right)$$

Electric Field

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q} \qquad \text{.......................................(2.2)}$$

The electric field intensity E at a point r (observation point) due a point charge Q located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \qquad \text{...(2.3)}$$

For a collection of N point charges $Q_1, Q_2, ...Q_N$ located at $\vec{r}_1, \vec{r}_2, ...\vec{r}_N$, the electric field intensity at point $\vec{r}\,'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3} \qquad \text{..(2.4)}$$

The expression (2.4) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \qquad \text{.............(2.5)}$$

When this expression is integrated over the source region, we get the electric field at the point P due to this distribution of charges. Thus the expression for the electric field at P can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} dv' \qquad \text{..(2.6)}$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} dl' \qquad \text{..(2.7)}$$

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} ds' \qquad \text{..(2.8)}$$

Electric flux density:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined

as:

$$\vec{D} = \varepsilon \vec{E} \quad \text{……………..(2.9)}$$

We define the electric flux *Y* as

$$\psi = \int_S \vec{D} . d\vec{s} \quad \text{....................................(2.10)}$$

Gauss's Law: Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

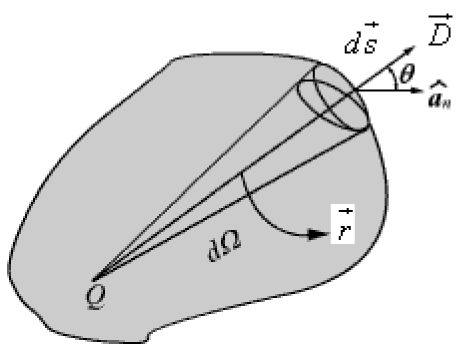


Fig 2.3: Gauss's Law

Application of Gauss's Law

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1.An infinite line charge

Let's consider the problem of determination of the electric field produced by an infinite line charge of density r_LC/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 2.4(a)

Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 2.4(b).
If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorem we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$ ....(2.11)

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero.

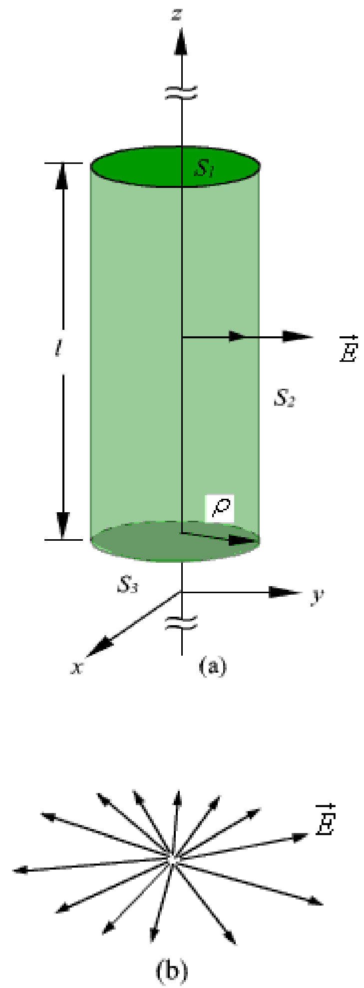


Fig 2.4: Infinite Line Charge

Hence we can write, $\rho_L l = \varepsilon_0 E.2\pi\rho l$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$ ....(2.12)

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x*-

z plane as shown in figure 2.5.

Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 2.5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y \qquad \text{.............(2.13)}$$

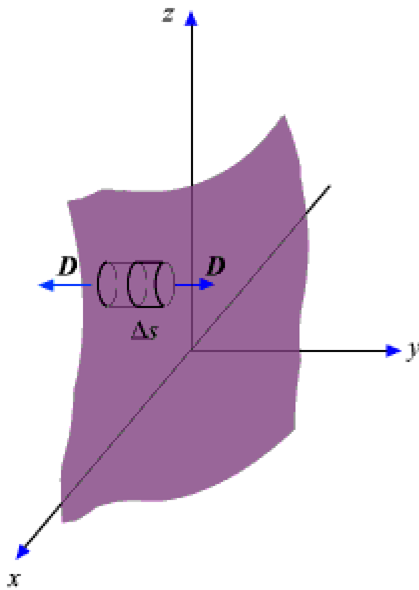


Fig 2.5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

Electrostatic Potential and Equipotential Surfaces

Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field. Let us suppose that we wish to move a positive test charge Δq from a point P to another point Q as shown in the Fig.2.8.

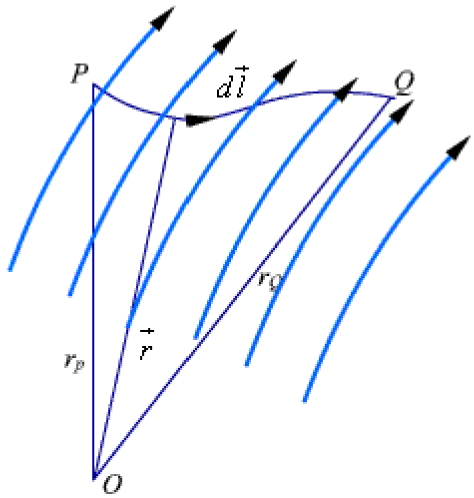


Fig 2.8: Movement of Test Charge in Electric Field

The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$$dW = -\Delta q \vec{E}.d\vec{l} \qquad \text{.............................(2.14)}$$

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \qquad \text{.....................................(2.15)}$$

The potential difference between two points P and Q , V_{PQ}, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \qquad \text{...............................(2.16)}$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as **Volts**.

Considering the movement of a unit positive test charge from an arbitrary point B to another arbitrary point A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E}\cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B \quad \text{..................................(2.17)}$$

So, the potential difference is independent of the path taken as it only depends on the initial & final points. It is customary to choose the potential to be zero at infinity. Thus potential at any point ($r_A = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $r_B = 0$).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \text{..................................(2.18)}$$

Or, in other words,

$$V = -\int_\infty^r E.dl \quad \text{..................................(2.19)}$$

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 2.9.

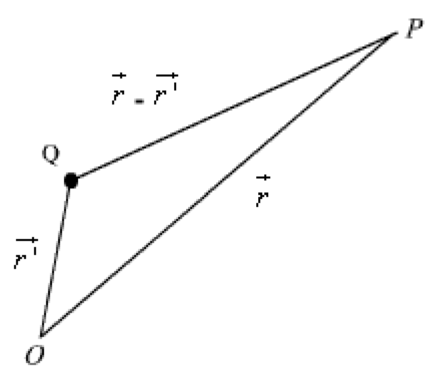


Fig 2.9: Electrostatic Potential due a Displaced Charge

The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r} - \vec{r}'\right|} \quad \text{..................................(2.20)}$$

Let us first consider N point charges Q_1, Q_2,.....Q_N located at points with position vectors $\vec{r}_1$, $\vec{r}_2$,...... $\vec{r}_N$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{|\vec{r}-\vec{r_1}|} + \frac{Q_2}{|\vec{r}-\vec{r_2}|} + \ldots\ldots \frac{Q_N}{|\vec{r}-\vec{r_N}|}\right) \quad \text{..................................(2.21a)}$$

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{|\vec{r}-\vec{r_n}|} \quad \text{...(2.21b)}$$

For continuous charge distribution, we replace point charges Q_n by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{|\vec{r}-\vec{r_n}|} \quad \text{..................................(2.22)}$$

For surface charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{|\vec{r}-\vec{r_n}|} \quad \text{.................................(2.23)}$$

For volume charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{|\vec{r}-\vec{r_n}|} \quad \text{.................................(2.24)}$$

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Since the potential difference is independent of the paths taken, V_{AB} = - V_{BA} , and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E}\cdot d\vec{l} = 0 \quad \text{.................................(2.25)}$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E}\cdot d\vec{l} = \int_S \left(\nabla\times\vec{E}\right)\cdot d\vec{s} = 0 \quad \text{............................(2.26)}$$

from which it follows that for electrostatic field,

$$\nabla\times\vec{E} = 0 \quad \text{..(2.27)}$$

Any vector field $\vec{A}$ that satisfies $\nabla\times\vec{A} = 0$ is called an irrotational field.

From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial x}dz = -\vec{E}\cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)\cdot\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E}\cdot d\vec{l}$$

$$\nabla V\cdot d\vec{l} = -\vec{E}\cdot d\vec{l} \quad \text{.................................(2.28)}$$

from which we obtain,

$$\vec{E} = -\nabla V \quad \text{..(2.29)}$$

Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure 2.10, the dipole is formed by the two point charges *Q* and -*Q* separated by a distance *d*, the charges being placed symmetrically about the origin. Let us consider a point *P* at a distance *r*, where we are interested to find the field.

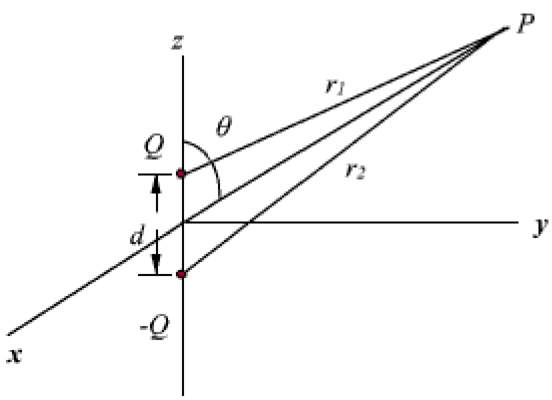


Fig 2.10 : Electric Dipole

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad \text{..........................(2.30)}$$

When r_1 and $r_2 >> d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and $r_1 \cong r_2 \cong r$.
Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \quad \text{..(2.31)}$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r \quad \text{..(2.32)}$$

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole
Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} \quad \text{................................(2.33)}$$

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance.

If the dipole is not centered at the origin, but the dipole center lies at $\vec{r}\,'$, the expression for the potential can be written as:

$$V = \frac{\vec{P}\cdot(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \quad \text{........................(2.34)}$$

The electric field for the dipole centered at the origin can be computed as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right) \quad \text{........................(2.35)}$$

$\vec{P} = Q\vec{d}$ is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r^3$ where as that of a point charge varies as $1/r^2$.

Work Done by the Electrostatic Field

• It is often useful to characterize any system that can impose forces on an object by the work it does to that object

• Suppose a charge q1 is located near another charge, q. The force acting on q causes work to be done by displacing q a distance dl.

$$dW = -\vec{F}\cdot d\vec{l} = -q\vec{E}\cdot d\vec{l}$$

• The negative sign indicates that the work done by an external agent, q1. Thus the total work done, or potential energy required) to move q a distance dl from a to b is:

$$W = -q\int_a^b \vec{E}\cdot d\vec{l} = (V(b) - V(a))q$$

Work and Energy in Electrostatic Fields

• To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them.

• Suppose we position 3 charges, q1, q2, and q3 in an initially empty space. Initially there is no work done to transfer charge q1 from infinity to our work space, because the space is initially charge free with no electric field in the region.

• However, there is a field present from q1 when we move q2 into position. The work done by transferring q2 into our workspace is the product of q2 times the potential difference between q1 and q2.

• The same is true as we position charge,q3, with respect to charges q1 and q2

$$W = W_1 + W_2 + W_3$$
$$W = 0 + q_1 V_{21} + q_3(V_{31} + V_{32})$$

If the charges were positioned in reverse order then:

$$W = W_1 + W_2 + W_3$$
$$W = 0 + q_2 V_{23} + q_3(V_{12} + V_{13})$$

• Thus by adding all of the work possibly performed we obtain

$$2W = 0 + q_2 V_{23} + q_3(V_{12} + V_{13}) + 0 + q_1 V_{21} + q_3(V_{31} + V_{32})$$
$$= q_1 V_1 + q_2 V_2 + q_3 V_3$$
$$W = \frac{1}{2}\sum_{k=1}^{n} q_k V_k$$

Therefore one can write that the energy, W, present in an electrostatic field fue to different charge distributions is:

$$W = \frac{1}{2}\int_L \varphi_L V dl \qquad W = \frac{1}{2}\int_S \varphi_S V dS \qquad W = \frac{1}{2}\int_V \varphi_V V dv$$

And since we can show by Gauss' Law that: $\varphi_v = \nabla \cdot \vec{D}$, so,

$$W = \frac{1}{2}\int_L \varphi_L V dl$$

$$W = \frac{1}{2}\int_v (\nabla \cdot \vec{D}) V dv$$

$$W - \frac{1}{2}\int_v \left((\nabla \cdot V\vec{D}) - (\vec{D} \cdot \nabla V)\right) dv$$

$$W = \frac{1}{2}\int_s V\vec{D} \cdot d\vec{S} - \frac{1}{2}\int_v \left(\vec{D} \cdot \nabla V\right) dv = -\frac{1}{2}\int_v \left(\vec{D} \cdot \nabla V\right) dv$$

($V\vec{D} \to \frac{1}{r^3}$, $d\vec{S} \to r^2$)

first integral → 0 as S becomes large

Energy $$W = \frac{1}{2}\int_v \left(\vec{D} \cdot \vec{E}\right) dv = \frac{1}{2}\int_v \varepsilon_o E^2 dv$$

Energy Density $$dW = w = \frac{\varepsilon_o E^2}{2} = \frac{D^2}{2\varepsilon_o}$$

QUESTIONS:

1.An electric field intensity is given as

$E = \frac{(100\cos\theta)}{r^3} a_r + \frac{(50 Sin\theta)}{r^3} a_\theta$; Calculate the |E| and a unit vector in Cartesian coordinate in the direction of E at a point (r=2, θ= 60^0 , ϕ=20^0)

2.Derive the expression for electric field due to two equal point charges of opposite sign(Electric dipole).

3.Give the basic Concepts of transformation of one coordinate system to another

4.Explain the Physical significance of term :

(i)Divergence of a vector

(ii)Curl of a vector field

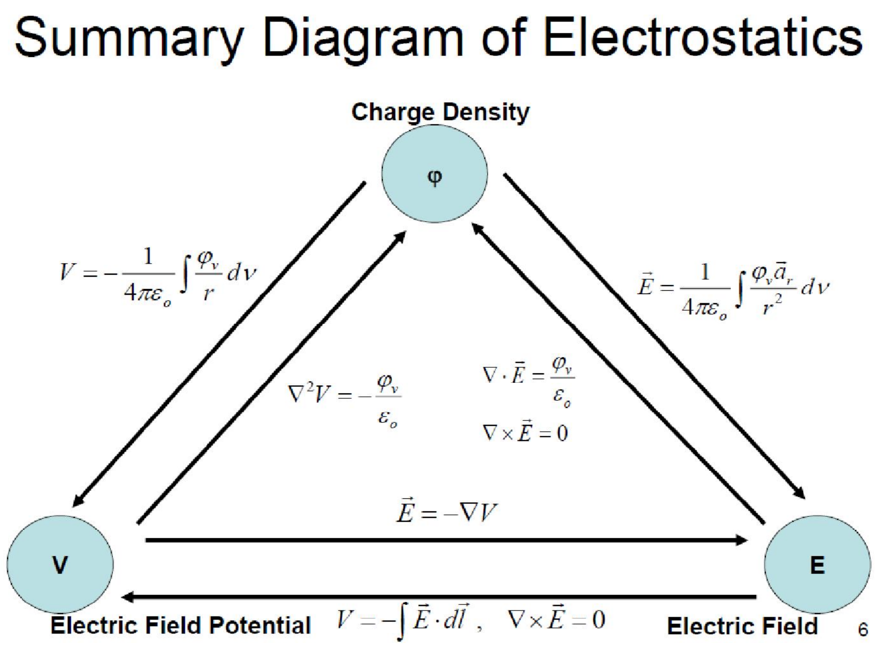
5.A Circular disk of radius R is charged to a uniform surface density ρ_s. Show that the electric field on the axis of the disk a distance x from the center is given by

$$E_x = \frac{\rho_s[1 - x(R^2 + x^2)^{-1/2}]}{2\varepsilon_0}$$

6.Considering a parallel plate capacitor, explain the concept of energy density.

7.In inkjet printer, quiet, fast printing on paper is accomplished by deflecting inkdroplets by electrostatic field. The arrangement is similar to that for a CRT except that a nozzle replaces the electron gun with droplets produced in a continuous stream. Charges are then sprayed onto the ejected droplets so that they can be electrostatically deflected as in a CRT. The nozzle of an ink-jet printer ejects droplets at 30m/s For a 4-mm deflection on the sheet being printed, find the deflecting field required if the deflecting field extent in the direction of the droplet's travel is 18mm. The sheet is 25mm from the leaving edge of the deflecting field. Assume an inkdrop mass of 40ng and charge of 250nC

SUMMARY:



MODULE-II

Currents & Conductors:

Convection and Conduction Currents:

• Current (in amperes) through a given area is the electric charge passing through the area per unit time,

$$\text{Current} \qquad I = \frac{dQ}{dt}$$

• Current density is the amount of current flowing through a surface, A/m2, or the current through a unit normal area at that point

$$\text{Current density} \qquad J - \frac{\Delta I}{\Delta S} \qquad \text{where} \quad I = \int_s \vec{J} \cdot d\vec{S}$$

• Depending on how the current is produced, there are different types of current density

– Convection current density

– Conduction current density

Convection Current Density

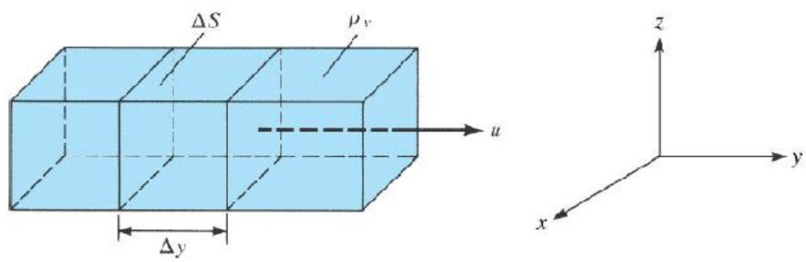
• Convection current density

– Does not involve conductors and does not obey Ohm's law

– Occurs when current flows through an insulating medium such as liquid, gas, or vacuum

$$\Delta I = \frac{\Delta Q}{\Delta t} = \varphi_v \Delta S \frac{\Delta y}{\Delta t} = \varphi_v \Delta S u_y$$

Where u is the velocity vector of the fluid.



$$J_y = \frac{\Delta I}{\Delta S} = \varphi_v u_y$$

$$\vec{J} = \varphi_v \vec{u}$$

Conduction Current Density

•Conduction current density

–Current in a conductor

–Obeys Ohm's law

•Consider a large number of free electrons traveling in a metal with mass (m), velocity (u), and scattering time (time between electron collisions), τ .

$$\vec{F} = -q\vec{E} = \frac{m\vec{u}}{\tau}$$

•The carrier density is determined by the number of electrons, n, with charge, e

$$\varphi_v = ne$$

•Conduction current density can then be calculate as

$$\vec{J} = \varphi_v \vec{u} = \frac{ne^2\tau}{m}\vec{E} = \sigma\vec{E}$$

•Where σ is the conductivity of the conductor

•This relationship between current concentration and electric field is known as Ohm's Law.

Electrical resistivity:

• Consider a conductor whose ends are maintained at a potential difference (i.e. the electric field within the conductor is nonzero and a field is passed through the material.)

• Note that there is no static equilibrium in this system. The conductor is being fed energy by the application of the electric field (bias potential)

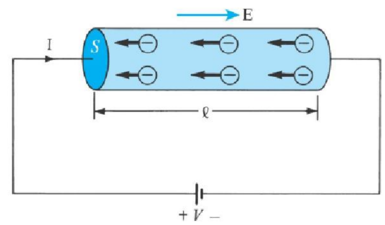
• As electrons move within the material to set up induction fields, they scatter and are therefore damped. This damping is quantified as the resistance, R, of the material.

• For this example assume:

– a uniform cross sectional area S, and length l.

– The direction of the electric field, E, produced is the same as the direction of flow of positive charges (or the same as the current, I).

$$R = \frac{V}{I} = \frac{\int_v \vec{E} \cdot d\vec{l}}{\int_s \sigma \vec{E} \cdot d\vec{S}}$$



So, we can write,

$$E = \frac{V}{l}$$

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{l}$$

$$R = \frac{V}{I} = \frac{l}{\sigma S} = \frac{\rho_c l}{S}$$

Continuity Equation

• Remembering that all charge is conserved, the time rate of decrease of charge within a given volume must be equal to the net outward flow through the surface of the volume.

• Thus, the current out of a closed surface is,

$$I = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ_{enclosed}}{dt} = \int_v -\frac{\partial \varphi_v}{\partial t} dv$$

Applying Stokes Theorem,

$$\oint_S \vec{J} \cdot d\vec{S} = \int_v \nabla \cdot \vec{J} dv = -\int_v \frac{\partial \varphi_v}{\partial t} dv$$

$$\nabla \cdot \vec{J} = -\frac{\partial \varphi_v}{\partial t}$$

The above equation is continuity equation.

QUESTIONS:

1.Show that Lorentz condition in the following equation is merely a restatement of continuity equation.

$$\nabla.A=-\mu\varepsilon\frac{dV}{dt}$$

2.A long copper wire of radius R runs through a deep lake at a height h above the plane bottom. Assuming the bottom to be a good conductor, show that the resistance per unit length between it and the wire is

$$\frac{Coh^{-1}(h/R)}{2\pi\sigma}$$ where σ= Conductivity of the lake water.

3.State the Continuity equation for steady currents

Polarization in Dielectrics

• The main difference between a conductor and a dielectric is the availability of free electrons in the outermost atomic shells to conduct current

• Carriers in a dielectric are bound by finite forces and as such, electric displacement occurs when external forces are applied

• Such displacements are produced when an applied electric field, E, creates dipoles within the media that polarize it

• Polarized media are evaluated by summing the original charge distribution and the dipole moment induced

• One may also define the polarization, P, of the material as the dipole moment per unit volume

$$\vec{P}=\lim_{\Delta v\to 0}\frac{\sum_{k=1}^{n}q_k\vec{d}_k}{\Delta v}=\lim_{\Delta v\to 0}\frac{\sum_{k=1}^{n}\vec{p}_k}{\Delta v}$$

• Potential due to a dipole moment

$$V=\frac{\vec{p}\cdot\vec{a}_r}{4\pi\varepsilon_o R^2}=\frac{\vec{p}}{4\pi\varepsilon_o}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$V=\int_v\frac{(\vec{p}\cdot\vec{a}_r)dv}{4\pi\varepsilon_o R^2}$$

Where,

$$R^2=|\vec{r}-\vec{r}\,'|^2=(x-x')^2+(y-y')^2+(z-z')^2$$

$$\nabla'\left(\frac{1}{R}\right)=-\frac{(x-x')\vec{a}_x+(y-y')\vec{a}_y+(z-z')\vec{a}_z}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}=-\frac{\vec{R}}{R^3}=\frac{\vec{a}_r}{R^2}$$

Now,

$$V = \int_v \frac{1}{4\pi\varepsilon_o}\left[\nabla'\left(\frac{\vec{P}}{R}\right) - \left(\frac{\nabla'.\vec{P}}{R}\right)\right]dv'$$

$$V = \int_v \frac{\vec{P}\cdot\vec{a}'_n}{4\pi\varepsilon_o R}dS' + \int_v \frac{-\nabla'.\vec{P}}{4\pi\varepsilon_o R}dv'$$

Where the Δ ' operator is with respect to (x',y',z').

$$\frac{\vec{P}\cdot\vec{a}_r}{R^2} = \vec{P}\cdot\nabla'\left(\frac{1}{R}\right) = \nabla'\left(\frac{\vec{P}}{R}\right) - \left(\frac{\nabla'.\vec{P}}{R}\right)$$

So, we can define two charge densities,

$$\boxed{\begin{aligned} \varphi_{ps} &= \vec{P}\cdot\vec{a}_n \\ \varphi_{pv} &= -\nabla\cdot\vec{P} \end{aligned}}$$

When polarization occurs, an equivalent volume charge density, φ_{pv}, is formed throughout the dielectric, while an equivalent surface charge density, φ_{ps}, is formed over the surface.

• For nonpolar dielectrics with no added free charge

$$Q_{total} = \oint_S \varphi_{ps} dS + \oint_v \varphi_{pv} dv = 0$$

• For cases in which the dielectric contains free charge density, φ_v

$$\varphi_t = \varphi_v + \varphi_{pv} = \nabla\cdot\varepsilon_o\vec{E}$$

Hence,

$$\begin{aligned} \varphi_v &= \nabla\cdot\varepsilon_o\vec{E} - \varphi_{pv} \\ &= \nabla\cdot\varepsilon_o\vec{E} + \nabla\cdot\vec{P} \\ &= \nabla\cdot(\varepsilon_o\vec{E} + \vec{P}) = \nabla\cdot\vec{D} \end{aligned}$$

• In many substances, experimental evidence shows that the polarization is proportional to the electric field, provided that E is not too strong. These substances are said to have a linear, isotropic dielectric constant.

• This proportionality constant is called the electric susceptibility, χ_e. The convention is to extract the permittivity of free space from the electric susceptibility to make the units dimensionless. Thus we have

$$\vec{P} = \varepsilon_o\chi_e\vec{E}$$

We know,

$$\vec{D} = \varepsilon_o\vec{E} + \vec{P} = \varepsilon_o(1+\chi_e)\vec{E}$$

$$\vec{D} = \varepsilon_o\varepsilon_r\vec{E}$$

$$\vec{D} = \varepsilon\vec{E}$$

Thus, the dielectric constant (or relative permittivity) of the material, ε_r, is the ratio of the permittivity to that of free space.

• If the electric field is too strong, then it begins to strip electrons completely from molecules leading to short term conduction of electrons within the media. This is called dielectric breakdown.

• The maximum strength of the electric field that a dielectric can tolerate prior to which breakdown occurs is called the dielectric strength.

• In **linear dielectrics**, the permittivity, ε , does not change with applied field, E.

• **Homogenous dielectrics** do not change their permittivity from point to point within the material.

• **Isotropic dielectrics** do not change their dielectric constant with respect to direction within the material.

• Two types of dielectrics exist in nature: polar and nonpolar

– Nonpolar dielectrics do not posses dipole moments until a strong electric field is applied

– Polar dielectrics such as water, posses permanent dipole moments that further align (if possible) in the presence of an external field

<u>Electric field in material medium:</u>

We have considered charge distributions only in free space & found it's effect in terms of electric field intensity, electric flux density & electrostatic potential. Now we'll consider effect of charge distributions in material medium.

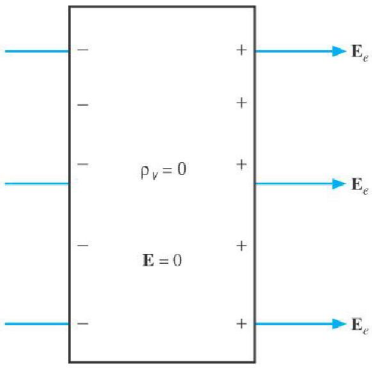
In general, based on the electric properties, materials can be classified into three categories: conductors, semiconductors and insulators (dielectrics). In *conductor* , electrons in the outermost shells of the atoms are very loosely held and they migrate easily from one atom to the other. Most metals belong to this group. The electrons in the atoms of *insulators* or *dielectrics* remain confined to their orbits and under normal circumstances they are not liberated under the influence of an externally applied field. The electrical properties of *semiconductors* fall between those of conductors and insulators since semiconductors have very few numbers of free charges. The parameter *conductivity* is used characterizes the macroscopic electrical property of a material medium.

Conductors

If some free charge is introduced inside a conductor, the charges will experience a force due to mutual repulsion and owing to the fact that they are free to move, the charges will appear on the surface. The charges will redistribute themselves in such a manner that the field within the conductor is zero. Therefore, under steady condition, inside a conductor $\rho_v = 0$, and using Gauss's theorem

$$\vec{E} = 0 \text{ ....(2.36)}$$

We know $\vec{E} = -\nabla V$, so a conductor behaves as an equipotential surface.



Boundary conditions:

Boundary conditions govern the behavior of electric fields at the boundary (interface) between two different media. The interface may be between two dielectrics or between a conductor & free space or between a conductor & dielectric. The last two cases will be special cases for first case.To complete this analysis we will use Gauss's theorem & conservative nature of electrostatic fields.

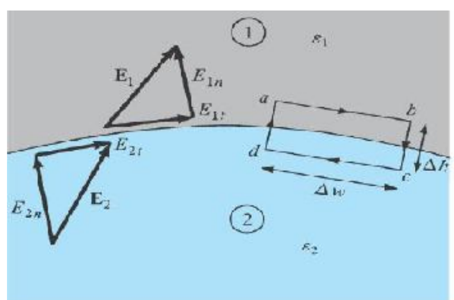
$$\nabla \cdot \vec{D} = \varphi_v \qquad \nabla \times \vec{E} = 0$$

• We will also need to break the electric field intensity into two orthogonal components (tangential and normal),

$$\vec{E} = \vec{E}_t + \vec{E}_n$$

Dielectric-Dielectric Boundary

• Two different dielectrics characterized by ε_1 and ε_2 .



Applying,

$$\nabla \times \bar{E} = 0 = \oint \bar{E} \cdot d\vec{l}$$

So,

$$\oint \bar{E} \cdot d\vec{l} = 0$$

$$= E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$$= E_{1t}\Delta w - E_{2t}\Delta w = (E_{1t} - E_{2t})\Delta w$$

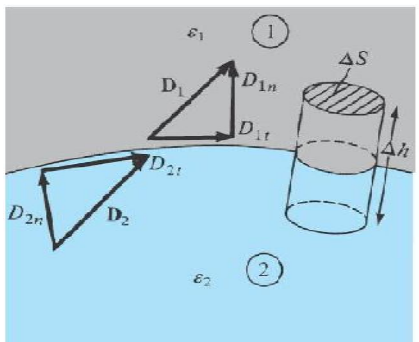
$$\Delta h \to 0$$

$$E_{1t} = E_{2t}$$

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

Thus, tangential E undergoes no change and is continuous across the boundary condition Tangential D on the other hand is discontinuous across the interface.

• Two different dielectrics characterized by ε_1 and ε_2 .



Applying,

$$\nabla \cdot \bar{D} = \varphi_v \Rightarrow \oint_S \bar{D} \cdot d\bar{S} = Q_{enc}$$

So,

$$\Delta Q = \varphi_s \Delta S = D_{1n}\Delta S - D_{2n}\Delta S$$

$$\varphi_s = D_{1n} - D_{2n}$$

$$\varphi_s \to 0$$

$$D_{1n} = D_{2n}$$

$$E_{1n}\varepsilon_1 = D_{1n} = D_{2n} = E_{2n}\varepsilon_2$$

Thus, normal D undergoes no change and is continuous across the boundary condition Normal E on the other hand is discontinuous across the interface.

So, we have,

$$E_{1t} = E_{2t}$$
$$D_{1n} = D_{2n}$$

Conductor-Dielectric Boundary

• Perfect conductor with infinite conductivity (therefore no volume charge density, potential or electric field inside the conductor) and a dielectric, ε_2 .

Apply $\oint \vec{E}\cdot d\vec{l} = 0$

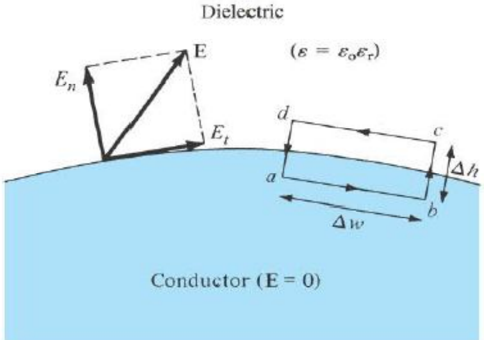
$$= 0\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_t\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

$$\Delta h \to 0, E_t = 0 = \frac{D_t}{\varepsilon_2}$$

Apply $\oint_S \vec{D}\cdot d\vec{S} = Q_{enc}$

$$\Delta Q = \varphi_s \Delta S = D_n \Delta S - 0\Delta S$$

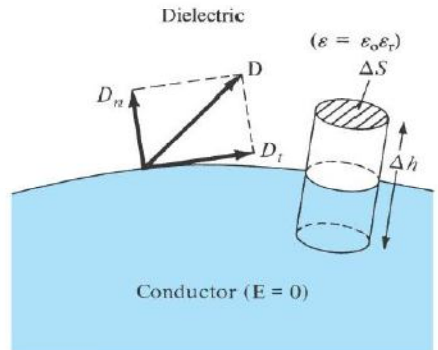
$$\varphi_s = D_n = \varepsilon_2 E_n$$



$$E_{2t} = 0$$

$$D_{2n} = \varepsilon_2 E_{2n}$$

$$E_1 = 0$$



Law of Refraction:

• Consider the boundary of two dielectrics, ε_1 and ε_2

• We can determine the refraction of the electric field across the interface using the dielectric boundary conditions provided.

$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\Delta h \to 0$$

$$E_{1t} = E_{2t}$$

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$$\oint_S \vec{D}\cdot d\vec{S} = Q_{enc}$$

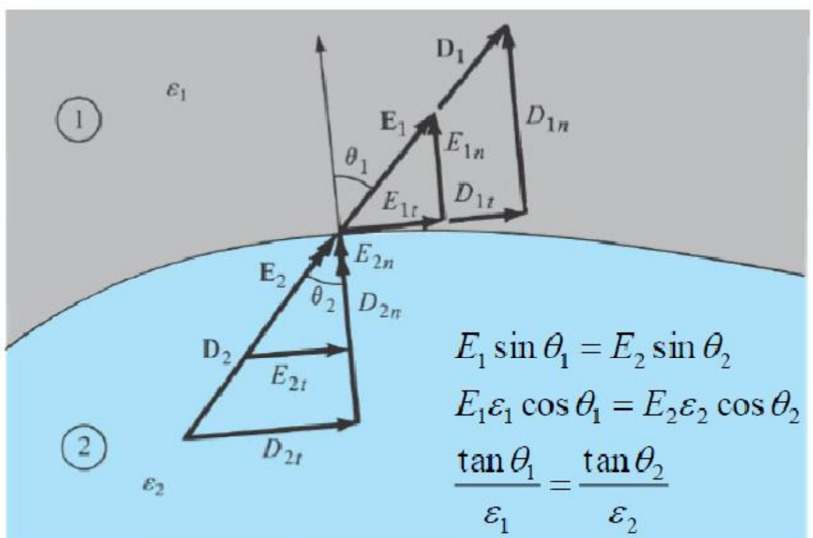
$$\varphi_s \to 0$$

$$D_{1n} = D_{2n}$$

$$E_{1n}\varepsilon_1 = D_{1n} = D_{2n} = E_{2n}\varepsilon_2$$

$$E_1\varepsilon_1 \cos\theta_1 = D_{1n} = D_{2n} = E_2\varepsilon_2 \cos\theta_2$$

$$E_1\varepsilon_1 \cos\theta_1 = E_2\varepsilon_2 \cos\theta_2$$



• Thus an interface between two dielectrics produces bending of flux lines as a result of unequal polarization charges that accumulate on the opposite sides of the interface.

<u>Electrostatic Boundary Value Problems</u>

Poisson's and Laplace's Equations for Electrostatics:

• Solving for the potential, V, using charge density.

$$\nabla \cdot \vec{D} = \nabla \cdot (\varepsilon \vec{E}) = \varphi_v$$

$$\vec{E} = -\nabla V$$

So,

$$-\nabla \cdot (\varepsilon \nabla V) = \varphi_v$$

$$\nabla^2 V = \frac{-\varphi_v}{\varepsilon}$$

The above equation is known as Poisson's equation. For a charge-free region the above equation can be written as,

$$\nabla^2 V = 0$$

The above equation is known as Laplace's equation.

• Uniqueness theorem: Although there are many ways to solve a differential equation, there is only one solution for any given set of boundary conditions.

Resistance:

For a uniform conductor, the resistance is given by,

$$R = \rho L/S$$

• We can also define it using Ohm's law, for a conductor with non-uniform cross-section, as,

$$R = \frac{V}{I} = \frac{\int \vec{E} \cdot d\vec{l}}{\oint \sigma \vec{E} \cdot d\vec{S}}$$

• The actual resistance in a conductor of non-uniform cross section can be solved as a boundary value problem using the following steps,

– Choose a coordinate system.

– Assume that V_o is the potential difference between two conductor terminals

– Solve Laplace's Eqn. to obtain V. Then Determine E = - ∇ V and solve I from

$$I = \int \sigma \vec{E} \cdot d\vec{S}$$

– Finally, R = Vo/I.

Capacitance

• Capacitance is the ratio of the magnitude of charge on two separated plates to the potential difference between them.

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \vec{E} \cdot d\vec{S}}{\int \vec{E} \cdot d\vec{l}}$$

The negative sign is dropped in the definition above because we are interested in the absolute value of the voltage drop.

• Capacitance is obtained by one of two methods

– Assuming Q, determine V in terms of Q

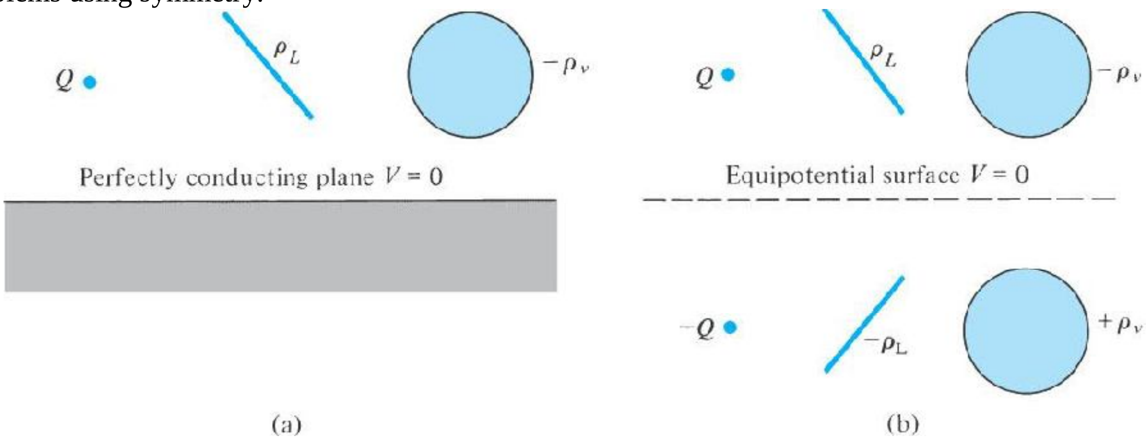
– Assuming V, determine Q in terms of V

- Using the above equation we can find the capacitance & resistance associated with parallel plate capacitor, coaxial cable & spherical capacitors.

Method of Images

•Image theory: A given charge configuration above an infinite grounded perfect conducting plane may be replaced by a mirror image of the charge configuration and an equipotential line in place of the conducting plane

•This theory is of significant importance because it allows one to significantly simplify complex problems using symmetry.



QUESTIONS:

1.Prove that any solution to Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used.
2.What is the current density of a convection current constituted by some charge in motion.
3.Derive the point form of continuity equation.
4.Define polarization of a dielectric. Establish the relationship between electric susceptibility, polarization vector and electric field intensity.
5.Why do charges remain on the surface of conductor.
6.What is electrostatic shielding? State the approach for finding the capacitance of a multiconductor system.
7.Two Condensers of capacity C_1 and C_2 possessing initially charges q_1 and q_2 respectively are connected in parallel. Show that there is a loss of electrostatic energy amounting to $\frac{(C_2q_1 - C_1q_2)^2}{2C_1C_2(C_1+C_2)}$. In what form does this energy appear?
8.Compute the work done in moving a point charge Q around a closed loop in a static field. What is the nature of electric field?

MODULE-III

Introduction to Magnetic Fields

Electrostatic fields are generated by static charges, magnetostatic fields are generated by static currents (charges that move with constant velocity in a particular direction).

•There are several similarities between electrostatic and magnetostatic fields

•For example, as we had E and D for electrostatics, we now use B and H to examine magnetic systems

•Our study of these fields allows us to evaluate and solve for a tremendous number of electric and electromechanical devices.

•Furthermore this study, will provide the basis for formulating an universal theory of Electromagnetic Fields that is utilized in almost every aspect of electrical engineering.

Analogy Between Electric and Magnetic Fields

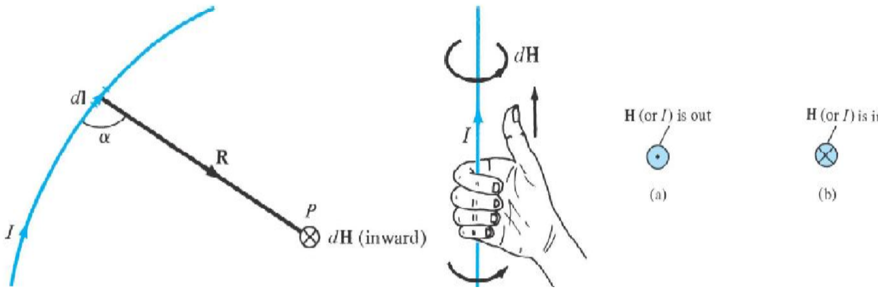
	Electric	**Magnetic**
• Basic Laws	$\vec{F} = \frac{Q_1 Q_1}{4\pi\varepsilon r^2}\hat{a}_r$	$d\vec{B} = \frac{\mu_0 I dl \times \hat{a}_r}{4\pi R^2}$
	$\oint \vec{D}\cdot d\vec{S} = Q_{enc}$	$\oint \vec{H}\cdot d\vec{l} = I_{enc}$
• Force Law	$\vec{F} = Q\vec{E}$	$\vec{F} = Q\vec{u}\times\vec{B}$
• Source Element	dQ	$Q\vec{u} = Id\vec{l}$
• Field intensity	$E = \frac{V}{l}(V/m)$	$H = \frac{I}{l}(A/m)$
• Flux density	$\vec{D} = \frac{\psi}{S}(C/m^2)$	$\vec{B} = \frac{\psi}{S}(Wb/m^2)$
• Relationship Between Fields	$\vec{D} = \varepsilon\vec{E}$	$\vec{B} = \mu\vec{H}$
• Potentials	$\vec{E} = -\nabla V$	$\vec{H} = -\nabla V_m, (\vec{J} = 0)$
	$V = \int \frac{\rho_L dl}{4\pi\varepsilon r}$	$A = \int \frac{\mu I d\vec{l}}{4\pi R}$
	$\psi = \oint \vec{D}\cdot d\vec{S}$	$\psi = \oint \vec{B}\cdot d\vec{S}$
• Flux	$\psi = Q = CV$	$\psi = LI$
	$I = C\frac{dV}{dt}$	$I = L\frac{dI}{dt}$
• Energy Density	$w_E = \frac{1}{2}\vec{D}\cdot\vec{E}$	$w_E = \frac{1}{2}\vec{B}\cdot\vec{H}$
• Poisson's Eqn.	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$	$\nabla^2\vec{A} = -\mu J$

4

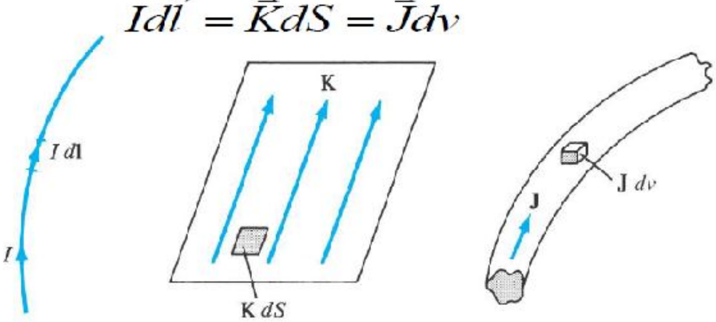
Biot-Savart's Law

• The differential magnetic field intensity, d**H**, produced at a point P, by the differential current element, Id**l**, is proportional to the product Id**l** and the sine of the angle between the element and the line joining P to the element and is inversely proportional to the square of the distance, R, between P and the element

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3} = \frac{Idl \sin\alpha}{4\pi R^2}$$



$$Id\vec{l} = \vec{K}dS = \vec{J}dv$$



Considering different current distributions(as shown above) we can rewrite expression for field intensity as below,

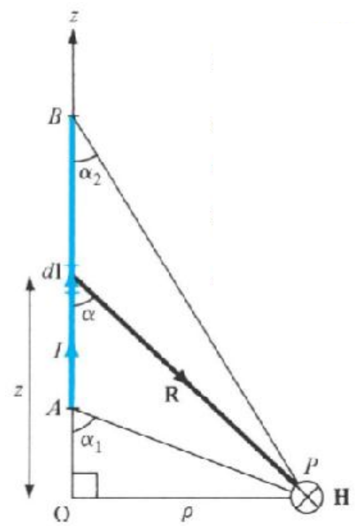
$$\vec{H} = \int_L \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2}$$

$$\vec{H} = \int_S \frac{\vec{K}dS \times \hat{a}_R}{4\pi R^2}$$

$$\vec{H} = \int_v \frac{\vec{J}dv \times \hat{a}_R}{4\pi R^2}$$

H Field From a Strait Current Carrying Filament

• The H field is determined for a strait filament of current in a manner very similar to that of the electric field determined from a line charge.



$$\vec{H} = \int_L \frac{I d\vec{l} \times \vec{R}}{4\pi R^3}$$

$$d\vec{l} = dz\hat{a}_z$$

$$\vec{R} = \rho\hat{a}_\rho - z\hat{a}_z$$

$$d\vec{l} \times \vec{R} = \rho dz\hat{a}_\phi$$

$$\vec{H} = \int_L \frac{I\rho dz\hat{a}_\phi}{4\pi(\rho^2 + z^2)^{3/2}}$$

Now,

$$z = \rho \cot \alpha$$

$$dz = -\rho(\csc^2 \alpha)d\alpha$$

$$\vec{H} = \int_L -\frac{I\rho^2(\csc^2 \alpha)d\alpha\hat{a}_\phi}{4\pi(\rho \csc \alpha)^3}$$

$$\vec{H} = -\frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}(\sin \alpha)d\alpha\hat{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi\rho}(\cos \alpha_2 - \cos \alpha_1)\hat{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi\rho}\hat{a}_\phi$$ Line from z = 0 to ∞

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$ Line from z = -∞ to ∞

Using Biot-savart's law we can find the expression for field intensity due to different current carrying conductor configurations.

Ampere's law:

The line integral of **H** around a closed path is the same as the net current, Ienc, enclosed by the path,

$$\oint \vec{H} \cdot d\vec{l} = I_{enc}$$

– Similar to Gauss' law since Ampere's law is easily used to determine H when the current distribution is symmetrical.

– Ampere's law ALWAYS holds, even if the current distribution is NOT symmetrical, however the equation is typically used for symmetric cases.

– Like Gauss and Coulomb's Laws, Ampere's law is a special case of the Biot-Savart law and can be derived directly from it.

• Applying Stokes's theorem, we have,

$$I_{enc} = \oint_L \vec{H} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{H}\right) \cdot d\vec{S}$$

We can also write using current density,

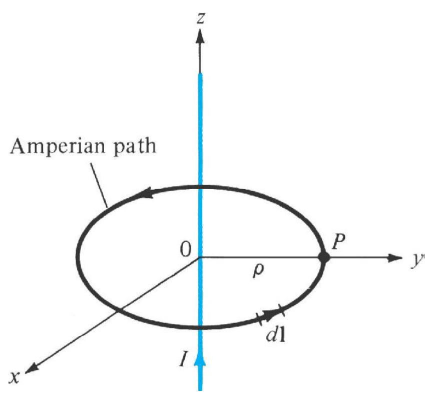
$$I_{enc} = \int_S \vec{J} \cdot d\vec{S}$$

So, from the above two equations,

$$\nabla \times \vec{H} = \vec{J}$$

Applications of Ampere's Circuit Law

• A simple application of Ampere's law can be used to easily derive the magnetic field intensity from an infinite line current ,



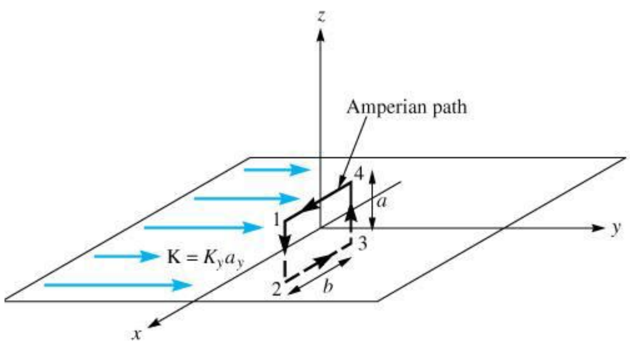
$$I_{enc} = \oint \vec{H} \cdot d\vec{l}$$

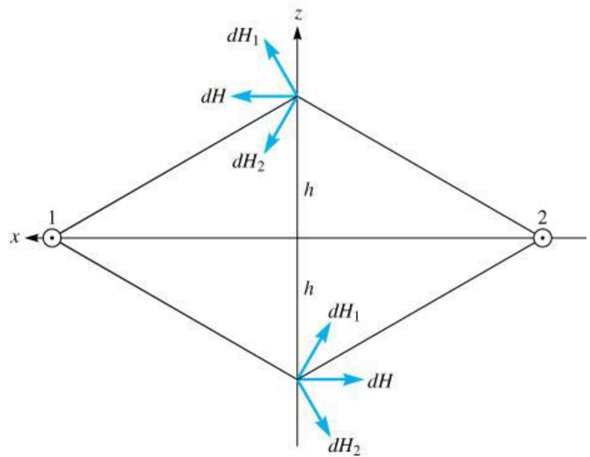
$$I = \int H_\phi \hat{a}_\phi \cdot \rho d\phi \hat{a}_\phi = H_\phi \int \rho d\phi = H_\phi \cdot 2\pi\rho$$

$$\vec{H} = \frac{I}{2\pi\rho} \hat{a}_\phi$$

Ampere's Circuit Law: Infinite Sheet of Current

• Consider an infinite sheet of current in the z=0 plane with a uniform current density, $\mathbf{K}=K_y\mathbf{a}_y$.





$$I_{enc} = \oint \vec{H} \cdot d\vec{l} = K_y b$$ Apply Ampere's law

$$\vec{H} = \begin{cases} H_0 \hat{a}_x, z > 0 \\ -H_0 \hat{a}_x, z < 0 \end{cases}$$

$$\oint \vec{H} \cdot d\vec{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \vec{H} \cdot d\vec{l}$$

$$= 0(-a) - H_0(-b) + 0(a) + H_0(b) = 2H_0 b$$

from Ampere's law and integral summation

$$H_o = K_y$$

$$\vec{H} = \begin{cases} \frac{1}{2} K_y \hat{a}_x, z > 0 \\ -\frac{1}{2} K_y \hat{a}_x, z < 0 \end{cases}$$

Thus for an infinite sheet of charge,

$$\vec{H} = \frac{1}{2} \vec{K} \times \hat{a}_n$$

So, by using Ampere's circuital law, the expressions for magnetic field intensity of different structures can be derived.

Magnetic Flux Density

• Magnetic Flux density, **B**, is the magnetic equivalent of the electric flux density, **D**. As such, one can define,

$$\vec{B} = \mu_0 \vec{H}$$

where $\mu_0 = 4\pi \times 10^{-7}\, H/m$

• Similarly, Ampere's Law is,

$$I_{enc} - \oint \frac{\vec{B}}{\mu_0} \cdot d\hat{l}$$

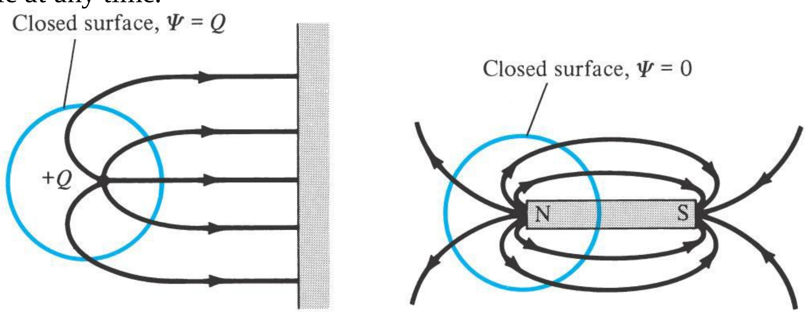
• And the Magnetic flux through a surface is,

$$\psi = \int_S \vec{B} \cdot d\vec{S} = \mu_0 \int_S \vec{H} \cdot d\vec{S}$$

• The magnetic flux through an enclosed system is,

$$\psi = \int_S \vec{B} \cdot d\vec{S} = \int_S \left(\nabla \cdot \vec{B}\right) dv$$

•Unlike electrostatic flux however, magnetic flux always follows a closed path and fold in on themselves. This simple statement has profound consequences. In electrostatics, we can easily define a point charge in which electric fields emanate to infinity. However, the solenoidal nature of the magnetic field requires magnetic flux to travel from a positive (north) to a negative (south) pole and it is not possible to have a single magnetic pole at any time.



(a) (b)

–There are NO magnetic monopoles, stipulating that an isolated magnetic charge DOES NOT EXIST

–The minimum field requirement for magnetics is a dipole.
So, mathematically,

$$\nabla \cdot \vec{B} = 0$$

Maxwell's Eqns. for Static Fields

Differential Form	Integral Form	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{S} - \int \rho_v dv$	Gauss's Law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	Nonexistence of the Magnetic Monopole
$\nabla \times \vec{E} = 0$	$\oint_L \vec{E} \cdot d\vec{l} = 0$	Conservative nature of the Electric Field
$\nabla \times \vec{H} = \vec{J}$	$\oint_L \vec{H} \cdot d\vec{l} - \oint_S \vec{J} \cdot d\vec{S}$	Ampere's Law

Magnetic Scalar & Vector Potential

We can define a magnetic field using the following requirements.

$$\nabla \times \nabla V = 0$$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

• Just as $\vec{E} = \;\; \nabla V$, we can define a magnetic scalar potential Vm related to **H** when the current density is zero as

$$\vec{H} - -\nabla V_m, \vec{J} - 0$$

$$\vec{J} = \nabla \times \vec{H} = \nabla \times \left(-\nabla V_m\right) = 0$$

$$\nabla^2 V_m = 0, \vec{J} = 0$$

• The requirement for a solenoidal field (and Maxwell's 4th law of electrostatics) stipulates

$$\nabla \cdot \vec{B} = 0$$

• And we can therefore define a magnetic vector potential, **A**, as

$$\vec{B} = \nabla \times \vec{A}$$

• Just as we defined the Electric Potential as $V = \int \frac{dQ}{4\pi\varepsilon_0 r}$, We can define the Magnetic Vector Potential as,

$$\vec{A} = \int_L \frac{\mu_0 I d\vec{l}}{4\pi R} \quad \text{for Line Current}$$

$$\vec{A} = \int_L \frac{\mu_0 \vec{K} dS}{4\pi R} \quad \text{for Surface Current}$$

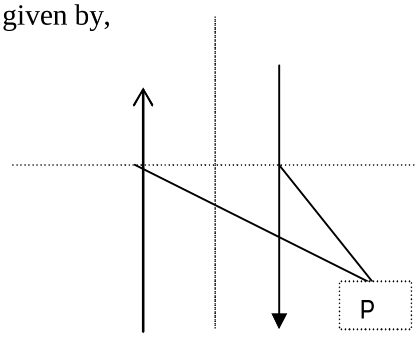
$$\vec{A} = \int_L \frac{\mu_0 \vec{J} dv}{4\pi R} \quad \text{for Volume Current}$$

<u>QUESTIONS:</u>

1.Derive Biot Savart law using the concept of vector magnetic potential.

2.Show that the vector Magnetic Potential A of two parallel infinite straight wires carrying current I in the opposite direction (as shown in Fig below) is given by,

$$A = \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_2}{r_1}\right) a_z$$



3.Calculate the force of repulsion per meter between two long parallel wires 30cm apart carrying current of 50amp in opposite directions

4.A circuit is in the form of a regular polygon of n sides inscribed in a circle of radius a. If it is carrying a current I, show that the magnetic induction at the center is given by

$\frac{\mu_0 ni}{2\pi a}\tan\frac{\pi}{n}$. Show that this expression approaches the induction at the center of a circle as n is indefinitely increased.

MODULE-III

Magnetic Forces Materials and Devices:

Lorentz Force Law:

• The force on a charged particle in an electric field is simply **F**=q**E**

• However, in the presence of an electromagnetic field an additional force is imposed from the charge displacement of velocity, **u**, quantified by the magnetic field, **B**.

• The combined force is defined by Lorentz Force Law:

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$

• Equating the Lorentz force to Newton's force equation, we have,

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) = m\vec{a} = m\frac{d\vec{u}}{dt}$$

Where 'a' is the acceleration of the particle is space.

Force on a Current Element

• One can also use field calculations to determine the force acting on a current element, Id**l**=**K**dS=**J**dv, due to an applied external magnetic field **B**.

• Assume that a copper wire carries a current density, **J**= ρ_v**u**.

• We know:

$$Id\vec{l} = \vec{J}dv = \rho_v u = dQu$$

• And that:

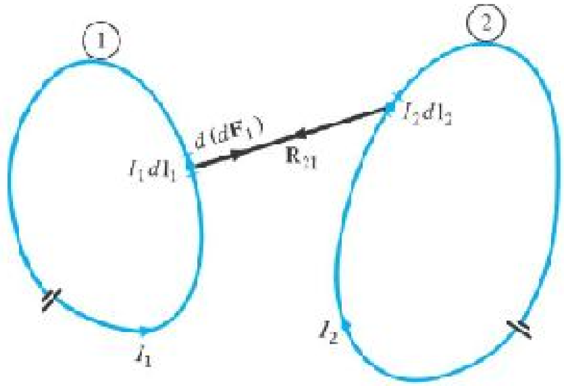
$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$
$$where \quad \vec{E} \rightarrow 0$$
$$\vec{F} = q(\vec{u} \times \vec{B})$$

Thus we can solve for the force on the first wire:

$$d\vec{F} = dq(\vec{u} \times \vec{B}) = Id\vec{l} \times \vec{B}$$
$$\vec{F} = \oint_L Id\vec{l} \times \vec{B}$$
$$Likewise$$
$$\vec{F} = \oint_L \vec{K}dS \times \vec{B} = \vec{F} = \oint_L \vec{J}dv \times \vec{B}$$

• Now one can define force on a current element from a magnetic field.

• However, that magnetic field must be generated somehow. What if it was generated by field produced from current passing through a second current element nearby. This means that currents in neighboring wires generate magnetic fields that generate forces on each other.



• Newton's law requires that the force, **F**1, acting on element 1 is equal and opposite to the force **F**2 acting on element 2.

• One can calculate these interdependent forces through the following derivation.

$$d\bar{F}_1 = I_1 d\bar{l}_1 \times \bar{B}_2$$

$$d(d\bar{F}_1) = I_1 d\vec{l}_1 \times d\bar{B}_2$$

From Biot- Savart's law we have,

$$d\bar{B}_2 = \frac{\mu_0 I_2 d\vec{l}_2 \times \hat{a}_R}{4\pi R_{12}^2}$$

On substituting,

$$d(d\bar{F}_1) = I_1 d\vec{l}_1 \times \frac{\mu_0 \left(I_2 d\vec{l}_2 \times \hat{a}_R\right)}{4\pi R_{12}^2}$$

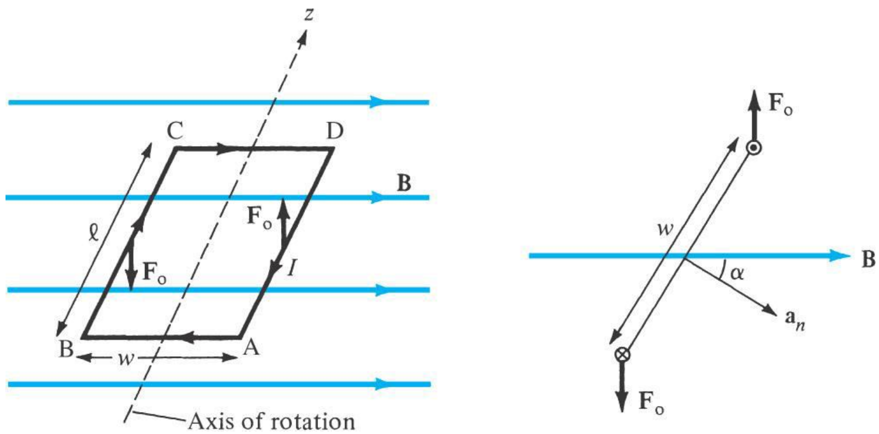
$$d\bar{F}_1 = \oint_{L_2} I_1 d\bar{l}_1 \times \frac{\mu_0 \left(I_2 d\vec{l}_2 \times \hat{a}_R\right)}{4\pi R_{12}^2}$$

$$\bar{F}_1 = \oint_{L_1}\oint_{L_2} I_1 d\bar{l}_1 \times \frac{\mu_0 \left(I_2 d\vec{l}_2 \times \hat{a}_R\right)}{4\pi R_{12}^2} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1}\oint_{L_2} d\bar{l}_1 \times \frac{\left(d\vec{l}_2 \times \hat{a}_R\right)}{R_{12}^2}$$

Using the above method we can find the force between two current carrying conductors.

Magnetic Torque and Moment:

• Let's examine the Torque applied to a current carrying loop.



• Torque, **T**, on the loop is the vector product of the force, **F**, and the moment arm, **r**.

$$\vec{T} = \vec{r} \times \vec{F}$$

$$|\vec{T}| = |\vec{r}||\vec{F}| \sin\alpha$$

And for a uniform magnetic field,

$$|\vec{F}_0| = IBl$$

$$|\vec{T}| = IBlw \sin\alpha$$

But, $lw = S$,so

$$|\vec{T}| = IBS \sin\alpha$$

Where we can now define a quantity **m** as the magnetic dipole moment with units A/m2 which is the product of the current and area of the loop in the direction normal the surface area defined by the loop

$$\vec{m} = IS\hat{a}_n$$

$$\vec{T} = \vec{m} \times \vec{B}$$

QUESTIONS:

1.A wire is bent into a plane to form a square of 30cm side and a current of 10 A is passed through it. Calculate H at the centre of the loop.

Magnetization in Materials:

• We know that all materials are made up of atoms consisting of electrons orbiting nuclei.

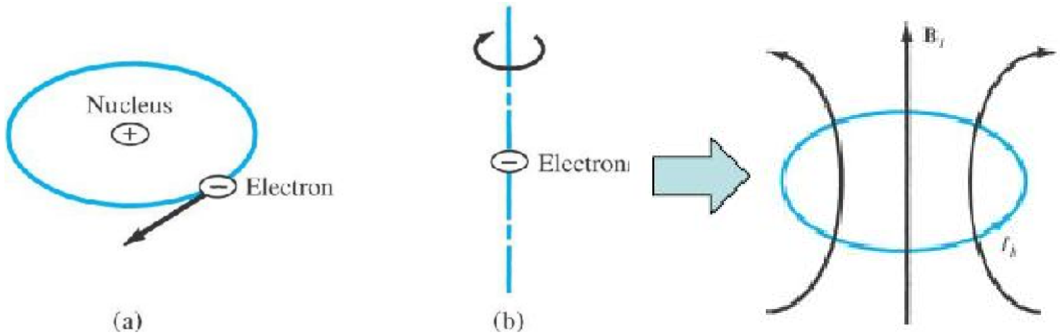
• Each of these electrons can also be said to spin about its axis.

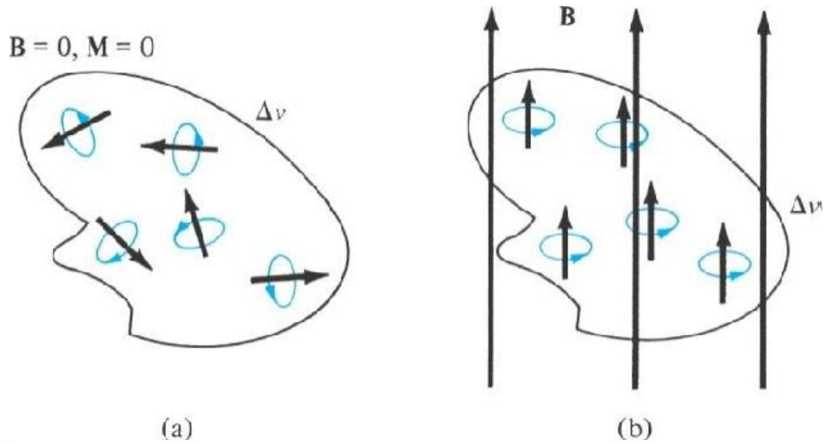
• In certain materials these spins associated with atomic magnetic dipoles align over large atomic distances to create magnetic domains across several thousands of atoms.

• As the individual magnetic domains align, over larger and larger volumes of the material, then the material is said to magnetize.

• Magnetization **M,** in A/m, is the magnetic dipole moment per unit volume.

• If N atoms are in a given volume, Δv, then the kth atom has a magnetic moment $\mathbf{m_k}$**.**





$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \vec{m}_k}{\Delta v}$$

$$d\vec{m} = \vec{M} dv'$$

$$d\vec{A} = \frac{\mu_0}{4\pi R^2} \left(\vec{M} \times \vec{a}_R \right) dv'$$

recalling

$$\frac{\vec{R}}{R^3} = \nabla' \frac{1}{R}$$

then

$$\vec{A} = \frac{\mu_0}{4\pi} \int \left(\vec{M} \times \nabla' \frac{1}{R} \right) dv'$$

We know,

$$\vec{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \vec{M} - \nabla' \times \frac{\vec{M}}{R}$$

So,

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\nabla' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi} \int_{v'} \nabla' \times \frac{\vec{M}}{R} dv'$$

Again,

$$\int_{v'} \nabla' \times \vec{F} dv' = -\oint_S \vec{F} \times d\vec{S}$$

So,

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\nabla' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M} \times d\vec{S}}{R} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\nabla' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M} \times \hat{a}_n}{R} dS$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\vec{J}_b dv'}{R} + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b dS'}{R}$$

Where b in the J and K terms represents a bound current densities

$$\vec{J}_b = \nabla \times \vec{M}$$

$$\vec{K}_b = \vec{M} \times \hat{a}_n$$

$$In\_free\_space, \Rightarrow \vec{M} = 0$$

$$\nabla \times \vec{H} = \vec{J}_f \Leftrightarrow \nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_f$$

$$However,\_in\_a\_material, \Rightarrow \vec{M} \neq 0$$

$$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_f + \vec{J}_b = \vec{J} = \nabla \times \vec{H} + \nabla \times \vec{M}$$

$$\Rightarrow \vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

For any linear magnetic material medium we have,

$$\vec{M} = \chi_m \vec{H}$$

So,

$$\vec{B} = \mu_0(1+\chi_m)\vec{H} = \mu_0\mu_r\vec{H} = \mu\vec{H}$$

yielding

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

Where μ is called the permeability of the material and is measured in H/m μr is called the relative permeability.

Classification of Magnetic Materials:

•In general we use the magnetic susceptibility (or relative permeability) to classify materials in terms of their magnetic property.

•A material is said to be nonmagnetic if there is no bound current density or zero susceptibility. Otherwise it is magnetic.

•Magnetic materials may be grouped into three classes: diamagnetic, paramagnetic, and ferromagnetic.

•For many practice purposes, diamagnetic and paramagnetic materials exhibit little to no magnetic. susceptibility. What magnetic properties these materials do have, follows a linear response over a large range of applied fields.

•Ferromagnetic materials kept below the Curie temperature exhibit very large nonlinear magnetic susceptibility and are used for conventional magnetic device applications.

Classification of Magnetic Materials

•Diamagnetism

–Occurs when the magnetic fields in the material due to individual electron moments cancels each other out. Thus the permanent magnetic moment of each atom is zero.

–Such materials are very weakly affected by magnetic fields.

–Diamagnetic materials include Copper, Bismuth, silicon, diamond, and sodium chloride (table salt)

–In general this effect is temperature independent. Thus, for example, there is no technique for magnetizing copper

–Superconductors exhibit perfect diamagnetism. The effect is so strong that magnetic fields applied across a superconductor do not penetrate more than a few atomic layers, resulting in B=0 within the material

•Paramagnetism

–Materials whose atoms exhibit a slight non-zero magnetic moment

–Paramangetism is temperature dependent

–Most materials (air, tungsten, potassium, monell) exhibit paramagnetic effects that provide slight magnetization in the presence of large fields at low temperatures

•Ferromagnetism

–Occurs in atoms with a relatively large magnetic moment

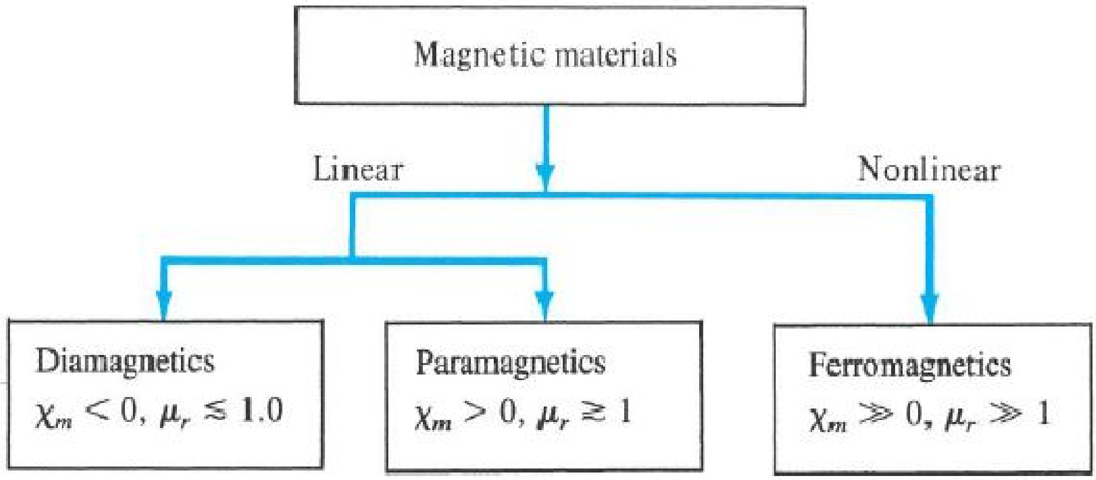
–Examples: Cobalt, Iron, Nickel, various alloys based on these three

–Capable of being magnetized very strongly by a magnetic field

–Retain a considerable amount of their magnetization when removed from the field

–Lose their ferromagnetic properties and become linear paramagnetic materials (non magnetic) when the temperature is raised above a critical temperature called the Curie temperature.

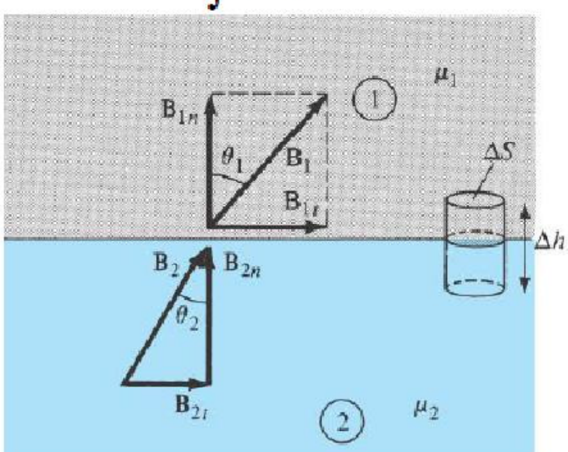
–Their magnetization is nonlinear. Thus the constitutive relation **B**= $\mu_0 \mu_r$ **H** does not hold because μ_r depends directly on **B** and cannot be represented by a single value.



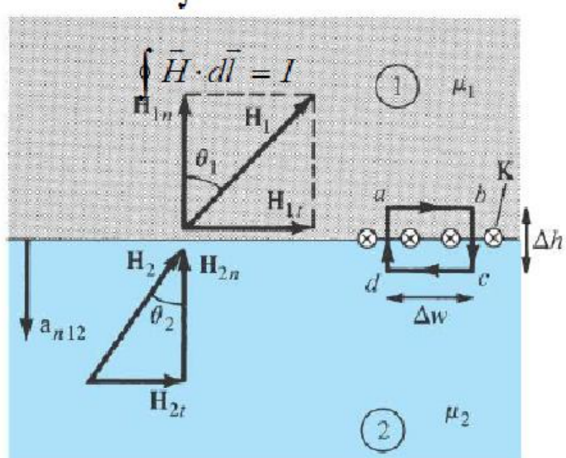
Magnetic Boundary Conditions:

• Magnetic boundary conditions for B and H crossing any material interface must match the following conditions developed using Guass's law for magnetic fields and Ampere's circuit law.

$$\oint \vec{B} \cdot d\vec{S} = 0$$



$$\oint \vec{H} \cdot d\vec{l} = I$$



$$\vec{B}_{1n} = \vec{B}_{2n}$$

$$\mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$$

$$\vec{H}_{1t} = \vec{H}_{2t}$$

$$\frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$$

•These boundary conditions can be used to develop an equivalent to Snell’s law for magnetic fields

$$B_1 \cos\theta_1 = |\vec{B}_{1n}| = |\vec{B}_{2n}| = B_2 \cos\theta_2$$

$$\frac{B_1}{\mu_1}\sin\theta_1 = |\vec{H}_{1t}| = |\vec{H}_{2t}| = \frac{B_{2t}}{\mu_2}\sin\theta_2$$

$$\mu_2 \tan\theta_1 = \mu_1 \tan\theta_2$$

Inductors and Inductance:

• We now know that closed magnetic circuit carrying current I produces a magnetic field with flux

$$\Psi = \int \vec{B}\cdot d\vec{S}$$

• We define the flux linkage between a circuit with N identical turns as,

$$\lambda = N\Psi$$

• As long as the medium the flux passes through is linear (isotropic) then then flux linkage is proportional to the current I producing it and can be written as,

$$\lambda = LI$$

Where L is a constant of proportionality called the inductance of the circuit. A circuit that contains inductance is said to be an inductor.

• One can equate the inductance to the magnetic flux of the circuit as

$$L = \frac{\lambda}{I} = \frac{N\Psi}{I}$$

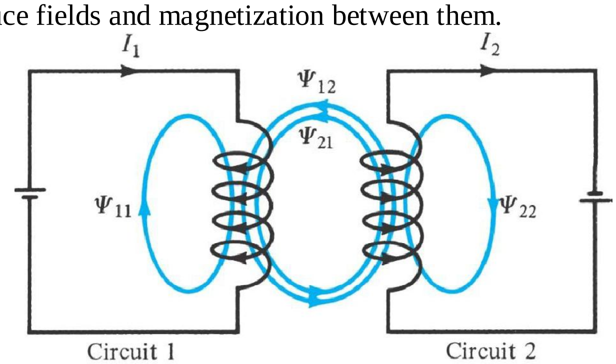
where L is measured in units of Henrys (H) = Wb/A.

• The magnetic energy (in Joules) stored by the inductor is expressed as,

$$W_m = \frac{1}{2}LI^2$$

Inductors and Inductance:

• Since we know that magnetic fields produce forces on nearby current elements, and that those magnetic fields can be generated by an isolated or coupled set of current carrying circuits, then it is only reasonable that such circuits may induce fields and magnetization between them.



• We can calculate the individual flux linkage between the two components as

$$\Psi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}$$

• Likewise we can determine a mutual inductance between the circuits that is equal from circuit 12 as it is from circuit 21 as

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \Psi_{12}}{I_2}$$

• Individual inductances are

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \Psi_1}{I_1} \qquad L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \Psi_2}{I_2}$$

• The total magnetic energy in the circuit is

$$W_m = W_1 + W_2 + W_{12} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2$$

• Mutual inductance may be calculated by the following method,

– Determine the internal inductance, Lin for the flux generated by the first inductor

– Determine the external inductance, Lext produced by the flux external of the first inductor

– The sum of the internal and external inductance equals the individual inductances plus the mutual inductance between the elements

$$M_{12} - \frac{\lambda_{12}}{I_2} - \frac{N_1 \Psi_{12}}{I_2} \qquad \Psi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}$$

Magnetic Energy

We can derive a similar term as derived for electric energy, for magnetic energy using the relation for energy as a function of inductance.

$$W_m = \frac{1}{2} L I^2$$

$$\Delta L = \frac{\Delta \Psi}{\Delta I} = \frac{\mu H \Delta x \Delta y}{\Delta I}$$

$$where,\ \Delta I = H \Delta z$$

$$\Delta L = \frac{\mu H \Delta x \Delta y}{\Delta I} = \frac{\mu \Delta x \Delta y}{\Delta z}$$

$$\Delta W_m = \frac{1}{2} \Delta L \Delta I^2 = \frac{1}{2} \mu H^2 \Delta x \Delta y \Delta z$$

$$\Delta W_m = \frac{1}{2} \mu H^2 \Delta v$$

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2 = \frac{1}{2} \left(\vec{H} \cdot \vec{B}\right) = \frac{B^2}{2\mu}$$

$$W_m = \int w_m dv = \int \frac{1}{2} \left(\vec{H} \cdot \vec{B}\right) dv = \int \frac{1}{2} \mu H^2 dv$$

Magnetic Circuits

•The following relations allow one to solve magnetic field problems in a manner similar to that of electronic circuits. It provides a clear means of designing transformers, motors, generators, and relays using a lumped circuit model. The analogy between electronic and magnetic circuits is provided below.

Table 8.4 Analogy between Electric and Magnetic Circuits

Electric	Magnetic
Conductivity σ	Permeability μ
Field intensity E	Field intensity H
Current $I = \int \mathbf{J} \cdot d\mathbf{S}$	Magnetic flux $\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
Current density $J = \frac{I}{S} = \sigma E$	Flux density $B = \frac{\Psi}{S} = \mu H$
Electromotive force (emf) V	Magnetomotive force (mmf) $\mathcal{F}$
Resistance R	Reluctance $\mathcal{R}$
Conductance $G = \frac{1}{R}$	Permeance $\mathcal{P} = \frac{1}{\mathcal{R}}$
Ohm's law $R = \frac{V}{I} = \frac{\ell}{\sigma S}$	Ohm's law $\mathcal{R} = \frac{\mathcal{F}}{\Psi} = \frac{\ell}{\mu S}$
or $V = E\ell = IR$	or $\mathcal{F} = H\ell = \Psi\mathcal{R} = NI$
Kirchhoff's laws: $\Sigma I = 0$, $\Sigma V - \Sigma RI = 0$	Kirchhoff's laws: $\Sigma \Psi = 0$, $\Sigma \mathcal{F} - \Sigma \mathcal{R} \Psi = 0$

Maxwell's Equations:

Faraday's Law for induced emf:

•Induced electromotive force (emf) (in volts) in any closed circuit is equal to the time rate of change of magnetic flux by the circuit,

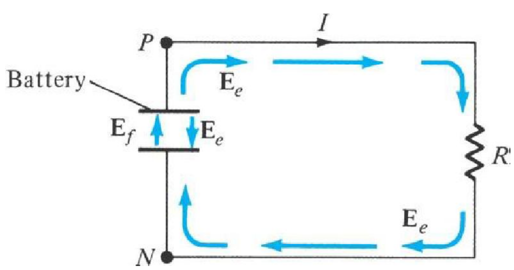
$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

where, as before, λ is the flux linkage, ψ is the magnetic flux, N is the number of turns in the inductor, and t represents a time interval. The negative sign shows that the induced voltage acts to oppose the flux producing it.

•The statement in blue above is known as Lenz's Law: the induced voltage acts to oppose the flux producing it.

•Examples of emf generated electric fields: electric generators, batteries, thermocouples, fuel cells, photovoltaic cells, transformers.

• To elaborate on emf, lets consider a battery circuit.



• The electrochemical action within the battery results and in emf produced electric field, **Ef**

• Acuminated charges at the terminals provide an electrostatic field **Ee** that also exist that counteracts the emf generated potential

$$\bar{E} = \bar{E}_f + \bar{E}_e$$

$$\oint_L \vec{E} \cdot d\vec{l} = \oint_L \vec{E}_f \cdot d\vec{l} + 0 = \int_N^P \vec{E}_f \cdot d\vec{l}$$

• The total emf generated in the between the two open terminals in the battery is therefore

$$V_{emf} = \int_N^P \vec{E}_f \cdot d\vec{l} = -\int_N^P \vec{E}_e \cdot d\vec{l} = IR$$

Transformer and Motional Electromotive Forces:

• The variation of flux with time may be caused by three ways

1. Having a stationary loop in a time-varying **B** field
2. Having a time-varying loop in a static **B** field
3. Having a time-varying loop in a time-varying **B** field

• A stationary loop in a time-varying **B** field

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

• A time-varying loop in a static **B** field

$$\nabla \times \vec{E}_m = \nabla \times (\vec{u} \times \vec{B})$$

• A time-varying loop in a time-varying **B** field

$$\nabla \times \vec{E}_m = -\frac{d\vec{B}}{dt} + \nabla \times (\vec{u} \times \vec{B})$$

Displacement Current:

• Lets now examine time dependent fields from the perspective on Ampere's Law.

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

This vector identity for the cross product is mathematically valid. However, it requires that the continuity eqn. equals zero, which is not valid from an electrostatics standpoint!

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

Thus, lets add an additional current density term to balance the electrostatic field requirement

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_d$$

$$\nabla \cdot \vec{J}_d = -\nabla \cdot \vec{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

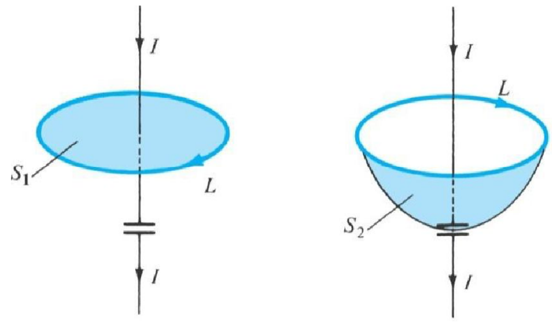
We can now define the displacement current density as the time derivative of the displacement vector

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Another of Maxwell's for time varying fields

This one relates Magnetic Field Intensity to conduction and displacement current densities

• We can apply the displacement current concept on the simple case of a capacitive element in a simple electronic circuit, as shown below.



Based on the equation for displacement current density, we can define the displacement current in a circuit as shown. Applying Ampere's circuit law to a closed path provides the following eqn. for current on the first side of the capacitive element. However surface 2 is the opposite side of the capacitor and has no conduction current allowing for no enclosed current at surface 2. If J =0 on the second surface then Jd must be generated on the second surface to create a time displaced current equal to current on surface 1.

$$\nabla \times \bar{H} - \bar{J} + \frac{\partial \vec{D}}{\partial t}$$

$$I_d = \int \bar{J} \cdot d\bar{S} = \int \frac{\partial \vec{D}}{\partial t} \cdot d\bar{S}$$

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_1} \vec{J} \cdot d\vec{S} = I_{enc} = I$$

$$\oint_L \bar{H} \cdot d\bar{l} = \int_{S_2} \vec{J} \cdot d\vec{S} = I_{enc} = 0$$

If J =0 on the second surface then J_d must be generated on the second surface to create a time displaced current equal to current on surface 1.

$$\oint_L \bar{\Pi} \cdot d\bar{l} - \int_{S_2} \vec{J}_d \cdot d\bar{S} - \frac{\partial}{\partial t} \int_{S_2} \vec{D} \cdot d\bar{S} - \frac{dQ}{dt} - I$$

We know,

$$D = \varepsilon E = \varepsilon \frac{V}{d}$$

So,

$$J_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d} \frac{dV}{dt}$$

$$\Rightarrow I_d = \vec{J}_d \cdot \vec{S} = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

$$from\_surface\_1$$

$$I_c = \frac{dQ}{dt} = S \frac{d\rho_s}{dt} = S \frac{dD}{dt} = \varepsilon S \frac{dE}{dt} = \frac{\varepsilon S}{d} \frac{dV}{dt} = C \frac{dV}{dt}$$

Maxwell's Time Dependent Equations

The Maxwell's equations for time dependent fields are,

Differential Form	Integral Form	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{S} = \int \rho_v dv$	Gauss's Law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$	Nonexistence of the Magnetic Monopole
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S}$	Faraday's Law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot d\vec{l} = \oint_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$	Ampere's Circuit Law

QUESTIONS:

1.Explain the terms "Self inductance" and "mutual inductance".

2.Derive the relationship between magnetic vector potential and current density vector

3.An electron travels with a velocity of 2×10^8 m/s perpendicular to a magnetic flux density of $0.15 W/m^2$. Determine the force on moving electron.

4.Draw a comparison between Electric and magnetic monopoles and dipoles.

5.Discuss the nature of various magnetic materials.

6.State the Maxwell's equations for static fields.

7.Show that the magnetic induction in Weber per square metre at the center of a square circuit of length l on a side carrying a current i is $\frac{2\sqrt{2}\mu_0 i}{\pi l}$ where i is in amperes and l is in meteres.

8.Write the expressions for vector magnetic potentials for three standard current configurations i.e. current filament, sheet current and volume current.

9.Derive Poisson's equation and also its analogous one for static magnetic field.

10.Prove that the field V=$\left(A\rho^4 + B\rho^{-4}\right)\sin 4\phi$ obeys Laplace's Equation

11. Prove that any solution to Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used.

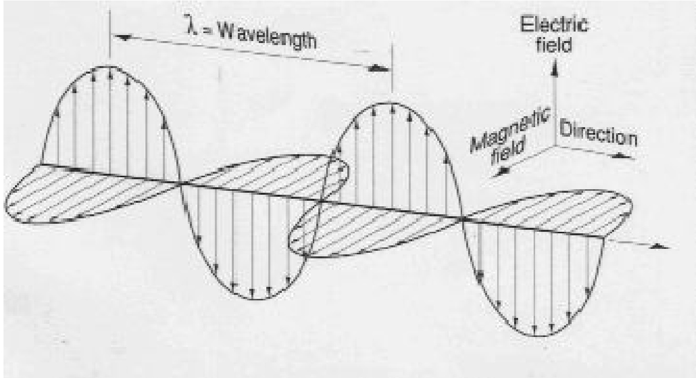
12.A 1-m diameter loop carries 25Amp. Find the magnetic flux density(B)

(i) at the center of the loop and (ii)on the loop axis 1m from the center

MODULE-IV

Plane Wave:

A uniform plane wave is the wave that the electric field, $\vec{E}$ or magnetic field, $\vec{H}$ in same direction, same magnitude and same phase in infinite planes perpendicular to the direction of propagation. A plane wave has no electric field, and magnetic field, components along its direction of propagation.



Wave Equations:

If the wave is in simple (linear, isotropic and homogeneous) nonconducting medium (σ =0), Maxwell's equation reduce to,

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \cdot \vec{H} = 0$$

The first-order differential equations in the two variables $\vec{E}$ and $\vec{H}$. They can combine to give $\vec{E}$ or $\vec{H}$ alone using second-order equation.

Using Maxwell's equation,

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad (1) \qquad \vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (2) \qquad \vec{\nabla} \cdot \vec{E} = 0 \quad (3)$$

The curl of equation of (1)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right)$$

Replacing in equation (2)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We know that $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla}^2 \vec{E}$ because of equation (3), thus the wave equation is

$$\vec{\nabla}^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The wave equation also can written as

$$\vec{\nabla}^2 \vec{E} - k^2 \vec{E} = 0$$ --------------------------(a)

Assuming an implicit time dependence $e^{j\omega t}$ in the field vector. Equation (a) also called Helmholtz equation. The *k* is called the wave number or propagation constant.

$$k - k_o \sqrt{\varepsilon_r} = \frac{2\pi f}{c} \sqrt{\varepsilon_r}$$

and

$$c - \frac{1}{\sqrt{\varepsilon\mu}}$$

where *c* is the velocity of light in free space.

For magnetic intensity domain, $\vec{H}$, we have,

$$\vec{\nabla}^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

or

$$\vec{\nabla}^2 \vec{H} - \mu_r \varepsilon_r k_o^2 \vec{H} = 0$$

For a uniform plane wave with an electric field $\vec{E} = \hat{x} E_x$ traveling in the *z*-direction, the wave equation can be reduced as

$$\frac{\partial^2 \vec{E}_x(z)}{\partial z^2} - k^2 \vec{E}_x(z) = 0$$

The solution of this wave equation,

$$\vec{E}(z) = \hat{x}E_x$$
$$= \hat{x}E_o e^{-kz}$$
$$= \hat{x}E_o e^{-\alpha z} e^{-j\beta z}$$

Where α is the attenuation constant of the medium and β is its phase constant.

The associated magnetic field, $\vec{H}$,

$$\vec{H}(z) = \hat{y}H_y$$
$$= \hat{y}\frac{\vec{E}_x}{\eta}$$
$$= \hat{y}\frac{E_o}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where η is the intrinsic impedance of the medium.
The *k* is called the wave number or propagation constant.

$$k^2 = k_o^2 \varepsilon_r \cdot \mu_r$$

$$k^2 = k_o^2 \mu_r (\varepsilon_r' - j\varepsilon_r'')$$

The wave number can also be written in terms of α and β.

$$k^2 = (\alpha + j\beta)^2$$
$$= (\alpha^2 - \beta^2) + j2\alpha\beta$$

Thus,

$$\alpha^2 - \beta^2 = k_o^2 \mu_r \varepsilon_r' \qquad \textbf{(1)}$$

$$2\alpha\beta = -k_o^2 \mu_r \varepsilon_r'' \qquad \textbf{(2)}$$

By solving (1) & (2),

$$\alpha = \sqrt{\frac{k_o^2 \mu_r \varepsilon_r'}{2}\left(\sqrt{1+\left(\frac{\varepsilon_r''}{\varepsilon_r'}\right)^2} - 1\right)}$$

$$\beta = \sqrt{\frac{k_o^2 \mu_r \varepsilon_r'}{2}\left(\sqrt{1+\left(\frac{\varepsilon_r''}{\varepsilon_r'}\right)^2} + 1\right)}$$

So for different medium,

Lossless Medium $(\sigma = 0)$	Low-loss Medium $(\varepsilon''/\varepsilon' \neq 0)$	Conductor $(\varepsilon''/\varepsilon' \to \infty)$
$\alpha = 0$	$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\alpha = \sqrt{\pi f \mu \sigma}$
$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \omega\sqrt{\mu\varepsilon}$	$\beta = \sqrt{\pi f \mu \sigma}$

Electromagnetic Phenomena are described by using four Maxwell's equations:

	Maxwell's equation		
		Description	**Information**
Gauss's Law (Electric fields)	**Integral form:** $\underbrace{\varepsilon_o \oint \vec{E} \cdot d\vec{S}}_{Left} = \underbrace{q}_{Right}$	**Left side:** The number of electric field lines – perpendicularly passing through to a closed surface, $\vec{S}$ **Right side:** Total amount of charge, q contained within that surface, .	Electric charge produces an electric field, $\vec{E}$ and the flux of that field passing through any closed surface is proportional to the total charge, q contained within that surface. Charge on an insulated conductor moves outward surface.
	Differential form: $\underbrace{\varepsilon_o \vec{\nabla} \cdot \vec{E}}_{Left} = \underbrace{\rho}_{Right}$	**Left side:** Divergence of the electric field, $\vec{E}$ – the tendency of the field to "flow" away from a specified location. **Right side:** Electric charge density, ρ	The electric field, $\vec{E}$ produced by electric charge diverges from positive charge and converges upon negative charge. The electric field, $\vec{E}$ is tendency to propagate perpendicularly away from a surface charge.

Gauss's Law **(Magnetic fields)**	**Integral form:** $\underbrace{\mu_o \oiint \vec{H}\cdot d\vec{S}}_{Left} = \underbrace{0}_{Right}$	**Left side:** The number of magnetic field lines – perpendicularly passing through a closed surface. **Right side:** Identically zero.	The total magnetic flux passing through any closed surface is zero. Flux enter the closed surface is same with the flux come out from the surface. The divergence of the magnetic field at any point is zero.
	Differential form: $\underbrace{\mu_o \vec{\nabla}\cdot\vec{H}}_{Left} = \underbrace{0}_{Right}$	**Left side:** Divergence of the magnetic field – the tendency of the field to "flow" away from a point than toward it. **Right side:** Identically zero.	

Faraday's Law	**Integral form:** $\underbrace{\oint_C \vec{E}\cdot d\vec{l}}_{Left} = \underbrace{-\mu_o \int_S \frac{\partial \vec{H}}{\partial t}\cdot d\vec{S}}_{Right}$	**Left side:** The circulation of the vector electric field, $\vec{E}$ around a closed path, C. **Right side:** The rate of change with time (d/dt) of magnetic field, through any surface, $\vec{S}$.	Changing magnetic flux through a surface induces an emf in any boundary path, C of that surface, and a changing magnetic field, $\vec{H}$ induces a circulating electric field.
	Differential form: $\underbrace{\vec{\nabla}\times\vec{E}}_{Left} = \underbrace{-\mu_o \frac{\partial \vec{H}}{\partial t}}_{Right}$	**Left side:** Curl of the electric field, – the tendency of the field lines to circulate around a point. **Right side:** The rate of change of the magnetic field, $\vec{H}$ over time (d/dt)	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time.

Ampere's Law	**Integral form:** $$\underbrace{\oint_C \vec{H}\cdot d\vec{l}}_{Left} = \underbrace{\int_S \left(\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}}_{Right}$$	**Left side:** The circulation of the magnetic field, $\vec{H}$ around a closed path, C. **Right side:** Two sources for the magnetic field, $\vec{H}$: a steady conduction current, $\vec{J}_c$ and a changing electric field, $\vec{E}$ through any surface, bounded by closed path, C.	An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path, C that bounds that surface.
	Differential form: $$\underbrace{\vec{\nabla}\times\vec{H}}_{Left} = \underbrace{\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}}_{Right}$$	**Left side:** Curl of the magnetic field, – the tendency of the field lines to circulate around a point. **Right side:** Two terms represent the electric current density, $\vec{J}_c$ and the time rate of change of the electric field, $\vec{E}$.	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time. An electric current, or a changing electric field, through a surface produces a circulating magnetic field, $\vec{H}$ around any path that bounds that surface.

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities associated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$

The above curl equations we can write

$$\vec{H}.\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) \quad , \quad \vec{E}.\frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right)$$

And $\vec{E}.\vec{J} = \sigma E^2$.

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV$$ ....(a)

The term $\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (a) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \text{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\hat{a}_x$$

when E_0 is real.

Let us consider two instanteneous quantities A and B such that

$A = \text{Re}\left(Ae^{j\omega t}\right) = |A|\cos(\omega t + \alpha)$, $B = \text{Re}\left(Be^{j\omega t}\right) = |B|\cos(\omega t + \beta)$

where A and B are the phasor quantities. i.e, $A = |A|e^{j\alpha}$

$B = |B|e^{j\beta}$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right]$$

Since A and B are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta)$$

Further, considering the phasor quantities *A* and *B*, we find that

$$AB^* = |A|e^{j\alpha}\,|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}(AB^*)$$

The poynting vector $\vec{P} = \vec{E}\times\vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x(E_yH_z - E_zH_y) + \hat{a}_y(E_zH_x - E_xH_z) + \hat{a}_z(E_xH_y - E_yH_x)$$ ....(b)

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (b) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6.41)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)\times H_y(z)\right)$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.
For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}(\vec{S})$.

Polarisation of plane wave:

The polarization of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficient as the magnetic field components are related to electric field vector by the Maxwell's equations.
Let us consider a plane wave travelling in the +z direction. The wave has both E_x and E_y components.

$$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$

The corresponding magnetic fields are given by,

$$\vec{H} = \frac{1}{\eta}\hat{a}_z \times \vec{E}$$

$$= \frac{1}{\eta}\hat{a}_z \times \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right) e^{-j\beta z}$$

$$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right) e^{-j\beta z}$$

Depending upon the values of E_{ox} and E_{oy} we can have several possibilities:

1. If $E_{oy} = 0$, then the wave is linearly polarised in the *x*-direction.
2. If $E_{oy} = 0$, then the wave is linearly polarised in the *y*-direction.
3. If E_{ox} and E_{oy} are both real (or complex with equal phase), once again we get a linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig 6.4.

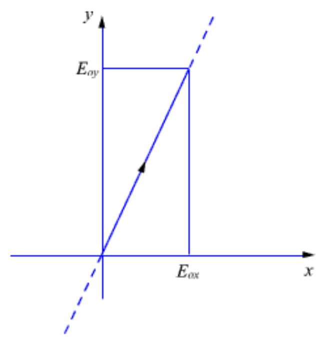


Fig 6.4 : Linear Polarisation

If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}| e^{ja}$, $E_{oy} = |E_{oy}| e^{jb}$

Then, $E_x(z,t) = \mathrm{Re}\left[|E_{ox}| e^{ja} e^{-j\beta z} e^{j\omega t}\right] = |E_{ox}|\cos(\omega t - \beta z + a)$

and $E_y(z,t) = \mathrm{Re}\left[|E_{oy}| e^{jb} e^{-j\beta z} e^{j\omega t}\right] = |E_{oy}|\cos(\omega t - \beta z + b)$ ....(c)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (c) we find that,

$$E_x(o,t) = |E_{ox}|\cos\omega t$$

$$E_y(o,t) = |E_{oy}|\cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin\omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1$$

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = |E_{ox}|\cos(\omega t)\hat{a}_x - |E_{oy}|\sin(\omega t)\hat{a}_y \quad \text{.........(d)}$$

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 6.5.

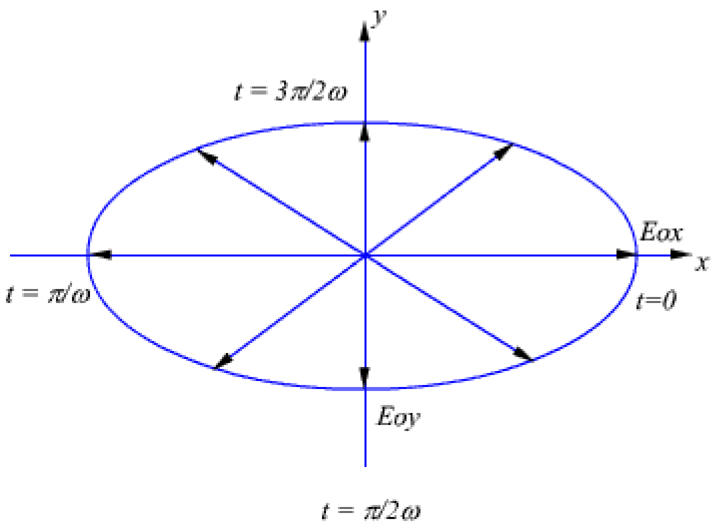


Figure 6.5 : Plot of *E*(*o*,*t*)

From equation (d) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces an ellipse and the field is said to be elliptically polarized.

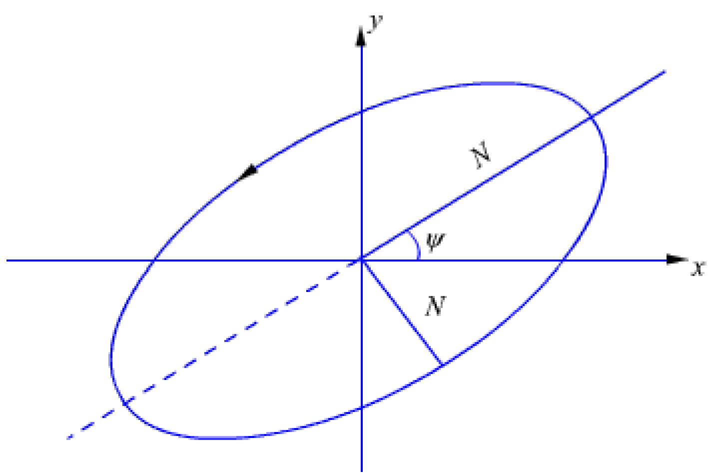


Figure 6.6: Polarisation ellipse

The polarisation ellipse shown in figure 6.6 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ (orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW). Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation (6.47), the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity.

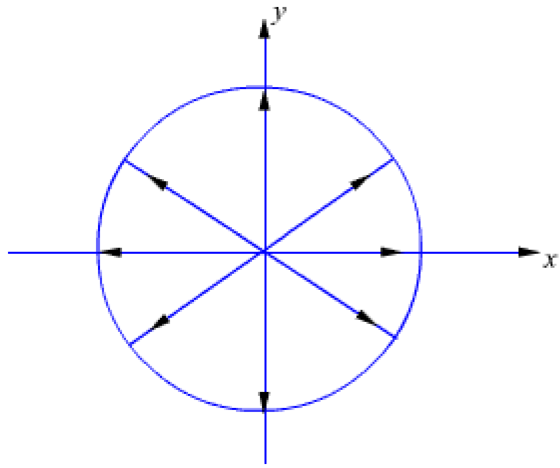


Figure 6.7: Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).

In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.
In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

Behaviour of Plane waves at the inteface of two media:
We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

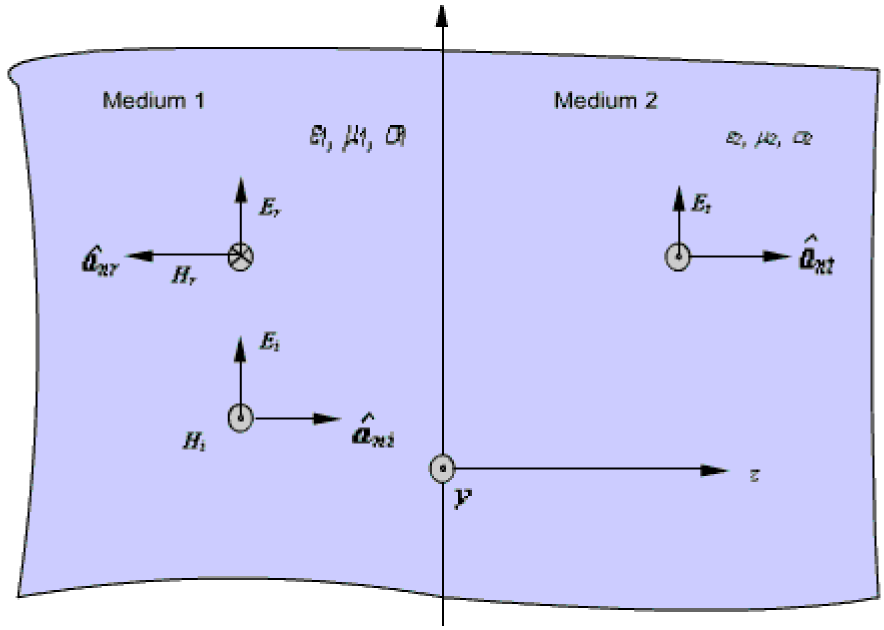


Fig 6.8 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.
Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively.

The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_X \quad \text{..................(e)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad \text{......................(f)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission. The reflected wave will travel along $\hat{a}_z$ in medium 1.
The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \quad \text{..(g)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \quad \text{.........(h)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \quad \text{..(i)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \qquad \text{...(j)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r \text{ and } \vec{H}_1 = \vec{H}_i + \vec{H}_r$$

and in medium 2,

$$\vec{E}_2 = \vec{E}_t \text{ and } \vec{H}_2 = \vec{H}_t$$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$$

$$\& \quad \vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

From equation (e) to (j) we get,

$$E_{io} + E_{ro} = E_{to} \qquad \text{...(k)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \qquad \text{..(l)}$$

Eliminating E_{to} ,

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}(E_{io} + E_{ro})$$

$$\text{or, } E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

$$\text{or, } E_{ro} = \tau E_{io}$$

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{...............(m)}$$

is called the reflection coefficient.

From equation (k) & (l), we can write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

$$\text{or, } E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$$

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$

is called the transmission coefficient.

We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$$

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \le |\tau| \le 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$

$$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$$

From (k) and (l)

$$\tau = -1$$

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

& $\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given $'t'$, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

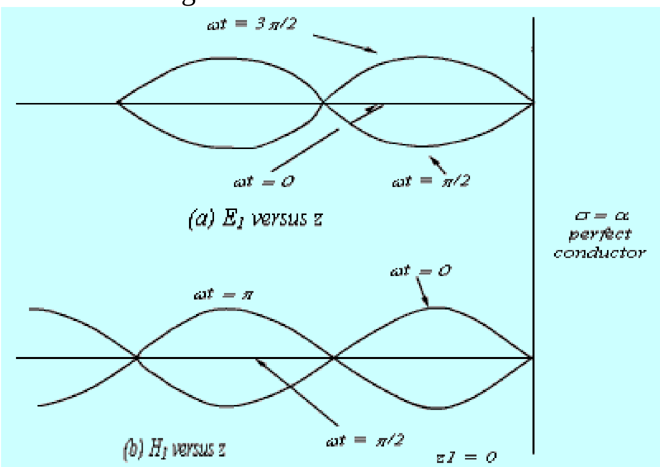


Figure 6.9: Generation of standing wave

Zeroes of $E_1(z,t)$ and

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4}, \ \ n = 0,1,2,....$$

zeroes of $H_1(z,t)$.

Case2: Normal incidence on a plane dielectric boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.
From above equations we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

In this case both η_1 and η_2 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right) \\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma(2j\sin\beta_1 z)\right)\end{aligned} \qquad ....(n)$$

From (n), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$.
The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows.
The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$
The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{\max} = E_{io}(1+T)$$

and this occurs when

$$2\beta_1 z_{\max} = -2n\pi$$

or $$z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$$, n = 0, 1, 2, 3......................(o)

The minimum value of $\left|\vec{E}_1\right|$ is

$$\left|\vec{E}_1\right|_{min} = E_{io}\left(1-\Gamma\right)$$ ....(p)

And this occurs when

$$2\beta_1 z_{min} = -\left(2n+1\right)\pi$$

or $$z_{min} = -\left(2n+1\right)\frac{\lambda_1}{4}$$, n = 0, 1, 2, 3............................(q)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1-\Gamma\right)$ which occurs at the z_{min} locations and the minimum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1+\Gamma\right)$ which occurs at z_{max} locations as given by the equations (o) and (q).

From our discussions so far we observe that $\frac{|E|_{max}}{|E|_{min}}$ can be written as

$$S = \frac{|E|_{max}}{|E|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

The quantity *S* is called as the standing wave ratio.

As $0 \le |\Gamma| \le 1$ the range of *S* is given by $1 \le S \le \infty$

We can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$$

From above equation we can see that $\left|\vec{H}_1\right|$ will be maximum at locations where $\left|\vec{E}_1\right|$ is minimum and vice versa.

In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface

So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor.
ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary

i. Perpendicular Polarization

The situation is depicted in figure 6.10.

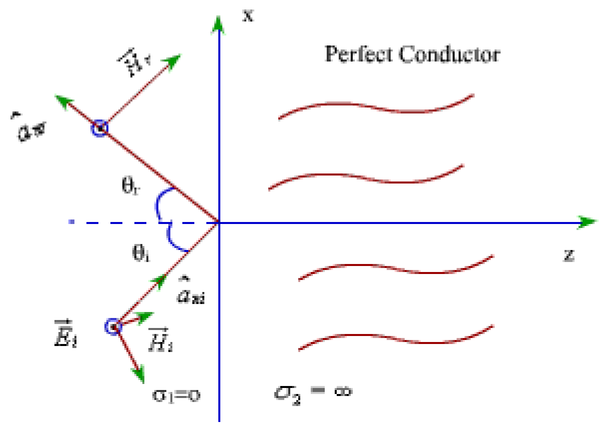


Figure 6.10: Perpendicular Polarization

As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. $\hat{a}_{ni}$ and $\hat{a}_{nr}$ respectively represent the unit vector in the direction of propagation of the incident and reflected waves, θ_i is the angle of incidence and θ_r is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_z \cos\theta_i + \hat{a}_x \sin\theta_i$$
$$\hat{a}_{nr} = -\hat{a}_z \cos\theta_r + \hat{a}_x \sin\theta_r$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component.

Therefore,

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1 \bar{a}_{ni}\cdot\bar{r}}$$
$$= \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The corresponding magnetic field is given by

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_n \times \vec{E}_i(x,z)\right]$$
$$= \frac{1}{\eta_1}\left[-\cos\theta_i \hat{a}_x + \sin\theta_i \hat{a}_z\right] E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Similarly, we can write the reflected waves as

$$\vec{E}_r(x,z) = \hat{a}_y E_{ro} e^{-j\beta_1 \bar{a}_{nr}\cdot\bar{r}}$$
$$= \hat{a}_y E_{ro} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
Since at the interface z=o, the tangential electric field is zero.
$$E_{io} e^{-j\beta_1 x\sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = 0$$
The above equation is satisfied if we have
$$E_{ro} = -E_{io}$$
$$\text{and } \theta_i = \theta_r$$

The condition $\theta_i = \theta_r$ is Snell's law of reflection.
$$\therefore \vec{E}_r(x,z) = -\hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$
$$\text{and } \vec{H}_r(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$$
$$= \frac{E_{io}}{\eta_1}\left[-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$
The total electric field is given by
$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$
$$= -\hat{a}_y 2jE_{io} \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$
Similarly, total magnetic field is given by
$$\vec{H}_1(x,z) = -2\frac{E_{io}}{\eta_1}\left[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} + \hat{a}_z j\sin\theta_i \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}\right]$$
From above two equations we observe that

1. Along z direction i.e. normal to the boundary

   y component of $\vec{E}$ and x component of $\vec{H}$ maintain standing wave patterns according to $\sin\beta_{1z}z$ and $\cos\beta_{1z}z$ where $\beta_{1z} = \beta_1\cos\theta_i$. No average power propagates along z as y component of $\vec{E}$ and x component of $\vec{H}$ are out of phase.
2. Along x i.e. parallel to the interface

   y component of $\vec{E}$ and z component of $\vec{H}$ are in phase (both time and space) and propagate with phase velocity

$$v_{p1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i}$$
$$\text{and } \lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i}$$

The wave propagating along the x direction has its amplitude varying with z and hence constitutes a **non uniform**plane wave. Further, only electric field is perpendicular to the direction of propagation (i.e. x), the magnetic field has component along the direction of propagation. Such waves are called transverse electric or TE waves.

ii. **Parallel Polarization**:

In this case also $\hat{a}_{ni}$ and $\hat{a}_{nr}$ are given by the derived equations. Here $\vec{H}_i$ and $\vec{H}_r$ have only y component.

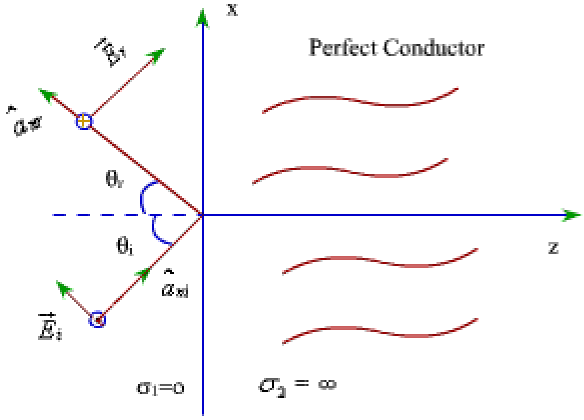


Figure 6.11: Parallel Polarization

With reference to fig (6.11), the field components can be written as:

Incident field components:

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \qquad \text{.............................(r)}$$

Reflected field components:

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Since the total tangential electric field component at the interface is zero.

$$E_i(x,0) + E(x,0) = 0$$

Which leads to $E_{io} = -E_{ro}$ and $\theta_i = \theta_r$ as before.

Substituting these quantities in (r) and adding the incident and reflected electric and magnetic field components the total electric and magnetic fields can be written as

$$\vec{E}_i(x,z) = -2E_{io}\left[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

and $$\vec{H}_i(x,z) = \hat{a}_y \frac{2E_{io}}{\eta_1}\cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}$$

Once again, we find a standing wave pattern along z for the x and y components of $\vec{E}$ and $\vec{H}$, while a non uniform plane wave propagates along x with a phase velocity given by $v_{p1x} = \frac{v_{p1}}{\sin\theta_i}$

where $v_{p1} = \frac{\omega}{\beta_1}$. Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves.

Oblique incidence at a plane dielectric interface

We continue our discussion on the behavior of plane waves at an interface; this time we consider a plane dielectric interface. As earlier, we consider the two specific cases, namely parallel and perpendicular polarization.

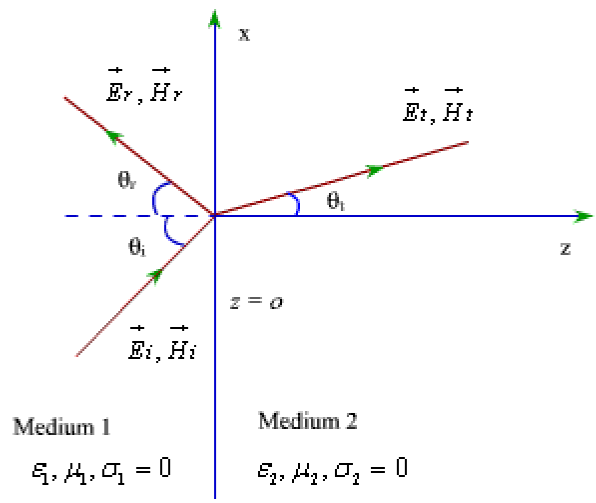


Fig 6.12: Oblique incidence at a plane dielectric interface

For the case of a plane dielectric interface, an incident wave will be reflected partially and transmitted partially.

In Fig(6.12), θ_i, θ_0 and θ_t corresponds respectively to the angle of incidence, reflection and transmission.

1. Parallel Polarization

As discussed previously, the incident and reflected field components can be written as

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

In terms of the reflection coefficient Γ

$$\vec{E}_r(x,z) = \Gamma E_{io}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{\Gamma E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

The transmitted filed can be written in terms of the transmission coefficient T

$$\vec{E}_t(x,z) = TE_{io}\left[\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t\right] e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \hat{a}_y \frac{TE_{io}}{n_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

We can now enforce the continuity of tangential field components at the boundary i.e. *z=0*

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = T\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } \frac{1}{n_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{n_1} e^{-j\beta_1 x\sin\theta_r} = \frac{T}{n_2} e^{-j\beta_2 x\sin\theta_t}$$ ....(s)

If both E_x and H_y are to be continuous at *z=0* for all *x* , then form the phase matching we have

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$\therefore$ We find that

$$\theta_i = \theta_r$$

$$\text{and } \beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$ ....(t)

Further, from equations (s) and (t) we have

$$\cos\theta_i + \Gamma\cos\theta_i = T\cos\theta_t$$

$$\text{and } \frac{1}{n_1} - \frac{\Gamma}{n_1} = \frac{T}{n_2}$$

$$\therefore \quad \cos\theta_i(1+\Gamma) = T\cos\theta_t$$

$$\text{and } \frac{1}{n_1}(1-\Gamma) = \frac{T}{n_2}$$

$$\therefore T = \frac{n_2}{n_1}(1-\Gamma)$$

$$\cos\theta_i(1+\Gamma) = \frac{n_2}{n_1}(1-\Gamma)\cos\theta_t$$

$$\therefore (n_1\cos\theta_i + n_2\cos\theta_t)\Gamma = n_2\cos\theta_t - n_1\cos\theta_i$$

or

$$\Gamma = \frac{n_2\cos\theta_t - n_1\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$

$$\text{and } T = \frac{n_2}{n_1}(1-\Gamma)$$

$$= \frac{2n_2\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$ ....(u)

From equation (u) we find that there exists specific angle $\theta_i = \theta_b$ for which Γ = *0* such that

$$n_2\cos\theta_t = n_1\cos\theta_b$$

or

$$\sqrt{1-\sin^2\theta_t} = \frac{n_1}{n_2}\sqrt{1-\sin^2\theta_b}$$

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

Further,

For non magnetic material $\mu_1 = \mu_2 = \mu_0$
Using this condition

$$1 - \sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_b\right)$$

$$\text{and } \sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_b$$ ....(v)

rom equation (v), solving for $\sin\theta_b$ we get

$$\sin\theta_b = \frac{1}{\sqrt{1 + \frac{\varepsilon_1}{\varepsilon_2}}}$$

This angle of incidence for which Γ = *0* is called Brewster angle. Since we are dealing with parallel polarization we represent this angle by $\theta_{b\|}$ so that

$$\sin\theta_{b\|} = \frac{1}{\sqrt{1 + \frac{\varepsilon_1}{\varepsilon_2}}}$$

2. **Perpendicular Polarization**

For this case

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{io}}{\eta_1}\left[-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right]e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_r(x,z) = \hat{a}_y \Gamma E_{io} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{\Gamma E_{io}}{\eta_1}\left[\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r\right]e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}_t(x,z) = \hat{a}_y T E_{io} e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{TE_{io}}{\eta_2}\left[-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t\right]e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

Using continuity of field components at *z=0*

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma e^{-j\beta_1 x\sin\theta_r} = TE_{io}e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } -\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = -\frac{T}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

As in the previous case

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

$$\therefore \quad \theta_i = \theta_r$$

$$\text{and } \sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

Using these conditions we can write

$$1 + \Gamma = T$$

$$-\frac{\cos\theta_i}{n_1} + \frac{\Gamma\cos\theta_i}{n_1} = -\frac{T\cos\theta_t}{n_2} \qquad \text{.........................(w)}$$

From equation (w) the reflection and transmission coefficients for the perpendicular polarization can be computed as

$$\Gamma = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}$$

$$\text{and } \quad T = \frac{2n_2\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}$$

We observe that if Γ = *0* for an angle of incidence $\theta_i = \theta_b$

$$n_2\cos\theta_b = n_1\cos\theta_t$$

$$\therefore \cos^2\theta_t = \frac{n_2}{n_1}\cos^2\theta_b$$

$$= \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\cos^2\theta_b$$

$$\therefore 1 - \sin^2\theta_t = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\left(1 - \sin^2\theta_b\right)$$

Again

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

$$\therefore \sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b$$

$$\therefore \left(1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b\right) = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\sin^2\theta_b$$

or

$$\sin^2\theta_b\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right) = \left(1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or

$$\sin^2\theta_b\left(\frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2\varepsilon_2}\right)\varepsilon_1 = \left(\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or

$$\sin^2\theta_b = \frac{\mu_2\left(\mu_1\varepsilon_2 - \mu_2\varepsilon_1\right)}{\varepsilon_1\left(\mu_1^2 - \mu_2^2\right)} \qquad \text{.........................(x)}$$

We observe if $\mu_1 = \mu_2 = \mu_0$ i.e. in this case of non magnetic material Brewster angle does not exist as the denominator or equation (x) becomes zero. Thus for perpendicular polarization in dielectric media, there is Brewster angle so that Γ can be made equal to zero.

From our previous discussion we observe that for both polarizations

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

If $\mu_1 = \mu_2 = \mu_0$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

For $\varepsilon_1 > \varepsilon_2$; $\theta_t > \theta_i$

The incidence angle $\theta_i = \theta_c$ for which $\theta_t = \frac{\pi}{2}$ i.e. $\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ is called the critical angle of incidence. If the angle of incidence is larger than θ_c total internal reflection occurs. For such case an evanescent wave exists along the interface in the x direction (w.r.t. fig (6.12)) that attenuates exponentially in the normal i.e. z direction. Such waves are tightly bound to the interface and are called surface waves.

QUESTIONS:

1.Write down Maxwell's field equations in the differential and integral form for time harmonic fields
2.Derive the expressions for energy stored in electric and magnetic field. Which field is efficient.

3.In a uniform plane wave, E and H are at right angles to each other. Prove.
4.A lossy dielectric is characterized by ε_R=1.5, μ_R=1 and $\sigma/\omega\varepsilon = 2.5\times10^{-4}$. At a frequency of 200MHz, how far can a uniform plane wave propagate in the material before

(i)it undergoes an attenuation 1Np

(ii)its amplitude is halved

5. Deduce the integral form of the theoram of Poynting and state the significance of the three terms appearing in the equation.

6.What are the properties of uniform plane wave?

7.Write Maxwell's equation in integral form and interpret

8.Show that characteristic impedance of free space is 377ohm

9.State and explain Poynting Vector(P) and Poynting theorem.

10.A brass(conductivity=10^7mho/m) pipe with inner and outer diameter of 3.4 and 4 cm carries a total current of 100A dc. Find Electric field (E), Magnetic field(H) and Poynting Vector(P) within the brass